

# Numerical Linear Algebra for Computational Science and Information Engineering CITHN2006

Final Exam

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## Problem 1 (9 pts)

Let  $A \in \mathbb{R}^{n \times n}$  be a non-singular matrix. We are interested in finding a right-approximate inverse of  $A$ , that is,  $M^{-1} \in \mathbb{R}^{n \times n}$  such that  $I_n - AM^{-1}$  is small in some sense.

- Show that  $(X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \mapsto \text{tr}(X^T Y)$  is an inner-product over  $\mathbb{R}^{n \times n}$ . (3 pts)
- Show that  $X \in \mathbb{R}^{n \times n} \mapsto \|X\|_F$  is a norm induced by the inner-product, i.e.,  $(X, X) = \|X\|_F^2$ . (1 pt)
- Consider the procedure described as follows:

$$\text{Find } M_1^{-1} \in M_0^{-1} + \text{span}\{G\} \text{ such that } R_1 := I_n - AM_1^{-1} \perp A \text{span}\{G\} \quad (1)$$

where  $M_0^{-1} \in \mathbb{R}^{n \times n}$  is an initial right-approximate inverse of  $A$ , and  $G \in \mathbb{R}^{n \times n}$  is a search direction. Assume  $AG \neq 0_{n \times n}$ , and find  $\alpha \in \mathbb{R}$  such that

$$M_1^{-1} = M_0^{-1} + \alpha G.$$

For the sake of brevity, introduce  $R_0 := I_n - AM_0^{-1}$ . (2 pts)

- Show that  $M_1^{-1}$  is given by Eq. (1) if and only if

$$M_1^{-1} = \arg \min_{M^{-1} \in M_0^{-1} + \text{span}\{G\}} \|I_n - AM^{-1}\|_F.$$

*Hint:* Use the orthogonal projection theorem. That is, for any  $\mathcal{S} \subset \mathbb{R}^{n \times n}$  and  $X \in \mathbb{R}^{n \times n}$ , there exists a unique  $Y \in \mathcal{S}$  such that  $\|X - Y\|_F = \min_{Z \in \mathcal{S}} \|X - Z\|_F$  if and only if  $X - Y \perp \mathcal{S}$ . (3 pts)

## Problem 2 (3 pts)

Consider a tall-and-skinny matrix  $A \in \mathbb{R}^{m \times n}$ , i.e., such that  $m \gg n$ , and let  $A = QR$  be the thin QR factorization of  $A$ , where  $Q \in \mathbb{R}^{m \times n}$  is orthogonal, and  $R \in \mathbb{R}^{n \times n}$  is upper-triangular.

- Write down the algorithm of the CholeskyQR method to compute the QR factorization of  $A$ . (2 pts)
- Is this algorithm stable? Explain why. (1 pt)

**Problem 3 (3 pts)**

Let  $A = \begin{bmatrix} 2 & 7 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 7 & 2 & 4 \\ 0 & 0 & 5 & 0 \end{bmatrix}$ , and answer the following questions with proper explanations:

- Is  $A$  singular? (1 pt)
- Does  $A$  admit an LU decomposition *without pivoting*? (2 pts)

**Problem 4 (5 pts)**

For the matrices

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

- Find the Rayleigh-Ritz pairs of  $A$  with respect to  $\text{range}(V)$ . (3 pts)
- Assemble the reduced eigenvalue problem to solve in order to find the harmonic Ritz values of  $A$  with respect to  $\text{range}(V)$  for a shift  $\sigma = 1$ . (2 pts)

**Problem 5 (9 pts)**

Let  $A = \begin{bmatrix} 1 & 4 & 5 & 6 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ , answer the following questions, and provide proper explanations:

- What is the spectrum of  $A$ ? (1 pt)
- Is  $A$  singular? (1 pt)
- Is  $A$  defective? (1 pt)
- Is  $A$  diagonalizable? (1 pt)
- Is  $A$  normal? (1 pt)
- What is the conditioning number of the smallest eigenvalue of  $A$ ? (3 pts)
- What is the conditioning number of each eigenvalue of  $B := A + A^T$ ? (1 pt)

**Problem 6 (3 pts)**

Complete the flowchart in Fig. 1 with the correct names of the methods covered in class:

- Conjugate gradient (CG),
- Minimal residual (MINRES),
- SYMMLQ,
- General minimal residual (GMRES),
- Quasi-minimal residual (QMR),
- Bi-conjugate gradient stabilized (Bi-CGSTAB)
- Conjugate gradient squared (CGS).

All the methods must be placed. Some boxes contain more than one method.

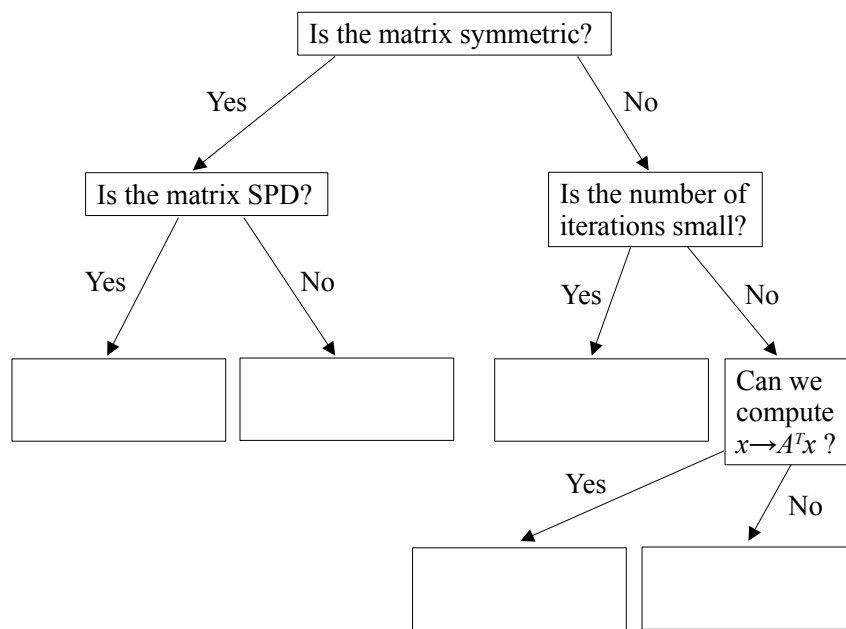


Figure 1: Flowchart of Krylov subspace-based linear iterative solvers.