Numerical Linear Algebra for Computational Science and Information Engineering
CITHN2006

Final Exam

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Problem 1 (2 pts)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

a. Is A normal, or not? Explain. (1 pt)

b. What is the conditioning number of solving for the (simple) smallest eigenvalue of A? (1 pt)

Problem 2 (4 pts)

Let $P \in \mathbb{R}^{n \times n}$ be any orthogonal projector, i.e., $P^T = P$. We show that P has a unit 2-norm, i.e., $\|P\|_2 = 1$, as follows:

a. Show that $\|x\|_2^2 = \|Px\|_2^2 + \|(I_n - P)x\|_2^2$ and $\|Px\|_2 \leq \|x\|_2 \, \forall \, x \in \mathbb{R}^n$. (2 pt)

b. Using the definition of the matrix 2-norm, show that $\|P\|_2 \geq 1$. (1 pt)

c. Using the result of a. with the definition of the matrix 2-norm, show that $\|P\|_2 \leq 1$. (1 pt)

Problem 3 (5 pts)

Assume A is a non-singular matrix and $\sigma \notin \mathrm{Sp}(A)$ is a given shift where $\mathrm{Sp}(A)$ denotes the spectrum of A.

a. Show that

$$\mathrm{Sp}(A) = \left\{ \sigma + \frac{1}{\vartheta}, \, \vartheta \in \mathrm{Sp}\left((A - \sigma I_n)^{-1}\right) \right\}$$

and

$$\min_{\lambda \in \mathrm{Sp}(A)} |\sigma - \lambda| = \sigma + \left(\max_{\vartheta \in \mathrm{Sp}((A - \sigma I_n)^{-1})} |\vartheta| \right)^{-1}.$$

(2 pts)

b. What is the shift-and-invert map $x \mapsto (A - \sigma I_n)^{-1} x$ used for when computing eigenvalues? Explain why. (2 pts)

c. Is it true or false that the harmonic Ritz vectors of A with respect to a search space $\mathcal{S}$ near a shift σ are obtained by applying a Rayleigh-Ritz procedure to the shift-and-invert operator $(A - \sigma I_n)^{-1}$ with respect to $(A - \sigma I_n)\mathcal{S}$ followed by one power iteration. (1 pt)

Problem 4 (5 pts)

a. Write down the pseudo-code for the Jacobi-Davidson method. (2 pts)

b. In the generalized Davidson method, an expansion vector is obtained by approximating the solution of a correction equation. What happens if this correction equation is solved exactly. Explain. (2 pts)

c. Is it true or false that when using Raleigh-Ritz approximations, a generalized Davidson iteration with the eigen-residual for expansion vector is equivalent to an Arnoldi-based procedure. (1 pt)

Problem 5 (3 pts)

Given a symmetric positive definite (SPD) matrix A and an initial iterate x_0, one definition of the conjugate gradient (CG) iterates is given by:

$$\begin{cases} x_{m+1} \in x_m + \text{span}\{p_m\} & \text{s.t. } \; r_{m+1} := b - Ax_{m+1} \perp \;\; \text{span}\{p_m\} \\ p_{m+1} \in r_{m+1} + \text{span}\{p_m\} & \text{s.t. } \; p_{m+1} \perp A\,\text{span}\{p_m\} \end{cases} \quad \text{for } m = 0, 1, 2, \ldots \tag{1}$$

where $r_0 := b - Ax_0$ and $p_0 := r_0$ denote the initial residual and search direction, respectively.

Show that, as long as the iterates have not converged, we have

$$x_{m+1} = x_m + \alpha_m p_m \;\; \text{where} \;\; \alpha_m = \frac{(r_m, r_m)}{(p_m, Ap_m)} \;\; \text{for} \;\; m = 0, 1, 2, \ldots$$

(3 pts)

Problem 6 (3 pts)

Complete the flowchart in Fig. 1 with the correct names of the methods covered in class:

- Conjugate gradient (CG),
- Minimal residual (MINRES),
- SYMMLQ,
- General minimal residual (GMRES),
- Quasi-minimal residual (QMR),
- Bi-conjugate gradient stabilized (Bi-CGSTAB)
- Conjugate gradient squared (CGS).

All the methods must be placed. Some boxes contain more than one method.

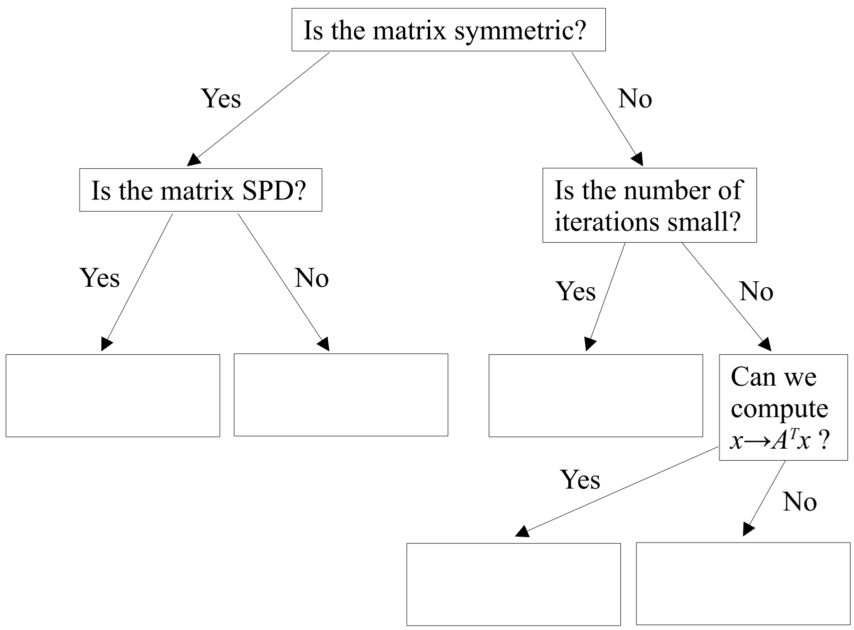

Figure 1: Flowchart of Krylov subspace-based linear iterative solvers.