Numerical Linear Algebra for Computational Science and Information Engineering

Lecture 16
Elements of Randomized Numerical Linear Algebra

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Outline

Introduction

- **Why** randomized numerical linear algebra (**RandNLA**)?
  - **Reducing algorithmic complexity** and **memory footprint**
  - **Improving** numerical **stability**
- **What** does RandNLA address?
  - **Mapping** of vectors and/or matrices onto **low dimensional spaces** using **randomized embeddings**, i.e., in a way that **approximately preserves** the **geometry** of transformed object **with high probability**.
- **What** is the **scope** of RandNLA?
  - Theory: Characterization of random embeddings, stability analysis, ...
  - Implementation: Fast sketching, ...
- Seminal works:
  - Least-squares problems
  - Compressed sensing, i.e., sparse signal recovery
  - Randomized trace estimation
  - Randomized SVD and randomized low-rank
  - Randomized Gram-Schmidt

Random subspace embeddings

(Deterministic) subspace embeddings

- Subspace embeddings can offer a means to reduce dimension of high-dimensional data.

Definition (Subspace embedding)

- A **subspace embedding** with **distortion** $\varepsilon \in (0, 1)$, or a ε-embedding, is a **linear map** $x \in \mathbb{F}^n \mapsto \Theta x \in \mathbb{F}^d$ which **embeds** $\mathbb{F}^d$ **into** $\mathcal{E} \subseteq \mathbb{F}^n$, i.e.,

$$(1 - \varepsilon)\|x\| \leq \|\Theta x\| \leq (1 + \varepsilon)\|x\| \; \forall \; x \in \mathcal{E}.$$

- Such maps, also referred to as **quasi-isometries**, require $d \geq \dim(\mathcal{E})$.

- In practice, we want to achieve **significant dimension reduction**, i.e., $d \ll n$, but this **limits the possible dimension of** $\mathcal{E}$.

(Deterministic) subspace embeddings, cont'd

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$$(1-\varepsilon)\|x\| \leq \|\Theta x\| \leq (1+\varepsilon)\|x\| \;\forall\, x \in \mathcal{E}.$$

- Such maps, also referred to as **quasi-isometries**, require $d \geq \dim(\mathcal{E})$.

- Randomization allows to *.

Randomized subspace embeddings

The limitation of d on the dimension of the embedding subspace $\mathcal{E} \subset \mathbb{F}^n$ is achieved by

Definition (Randomized subspace embedding)

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$$\Pr\left\{(1-\varepsilon)\|x\| \leq \|\Theta x\| \leq (1+\varepsilon)\|x\|\right\} \geq 1-\delta$$

- Such maps, also referred to as **quasi-isometries**, require $d \geq \dim(\mathcal{E})$.

- stuff

Sketching methods

Content

- Counting sketch
- SASO
- Subsampled trigonometric transforms

Randomized low-rank

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