Homogenization based on realization-dependent Hashin-Shtrikman functionals of piecewise polynomial trial polarization fields

N. Venkovic
nvenkov1@jhu.edu

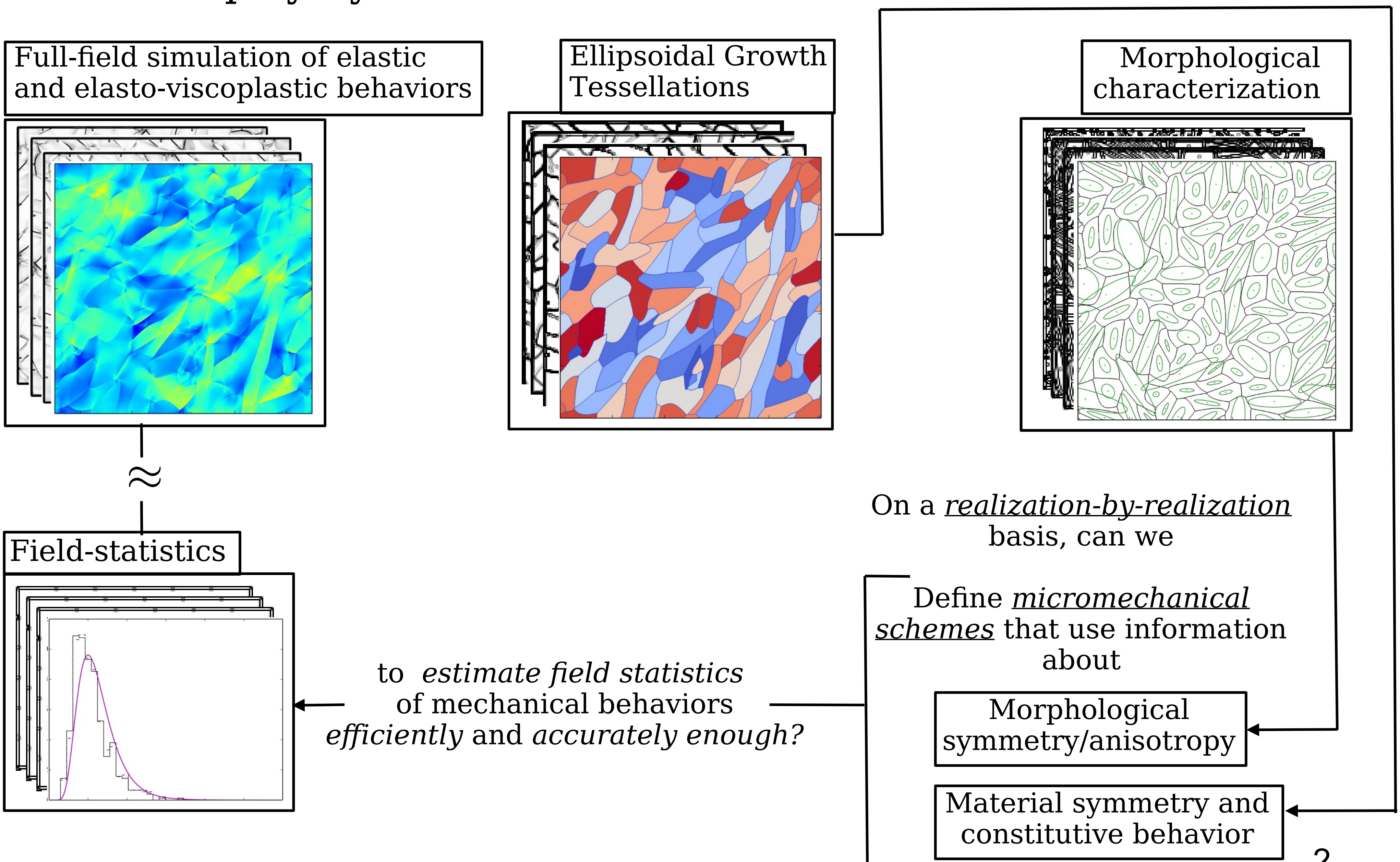
Supervisor: Prof. L. Graham-Brady
lori@jhu.edu

Group Meeting

October 28, 2017

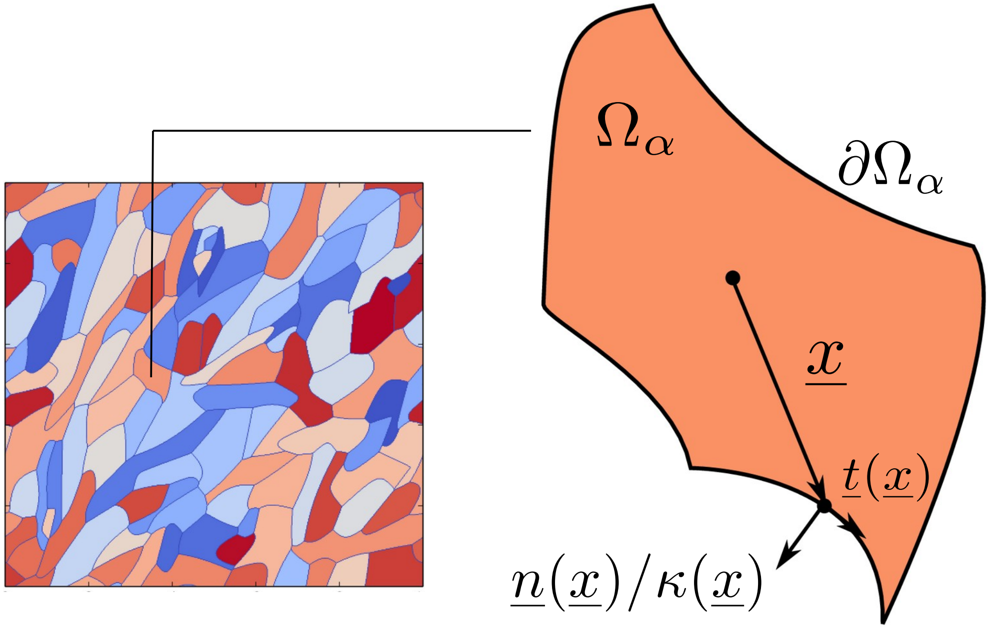
Motivation/Objective

- Understand the role of morphology on the mechanical performance of random polycrystals



Morphological characterization

Single grains are characterized using Minkowski tensors:



<u>Measures of mass distribution:</u>

$$\mathcal{W}_0^{r,0} = \int_{\Omega_\alpha} \underline{x}^{\otimes^r} \mathrm{d}V$$

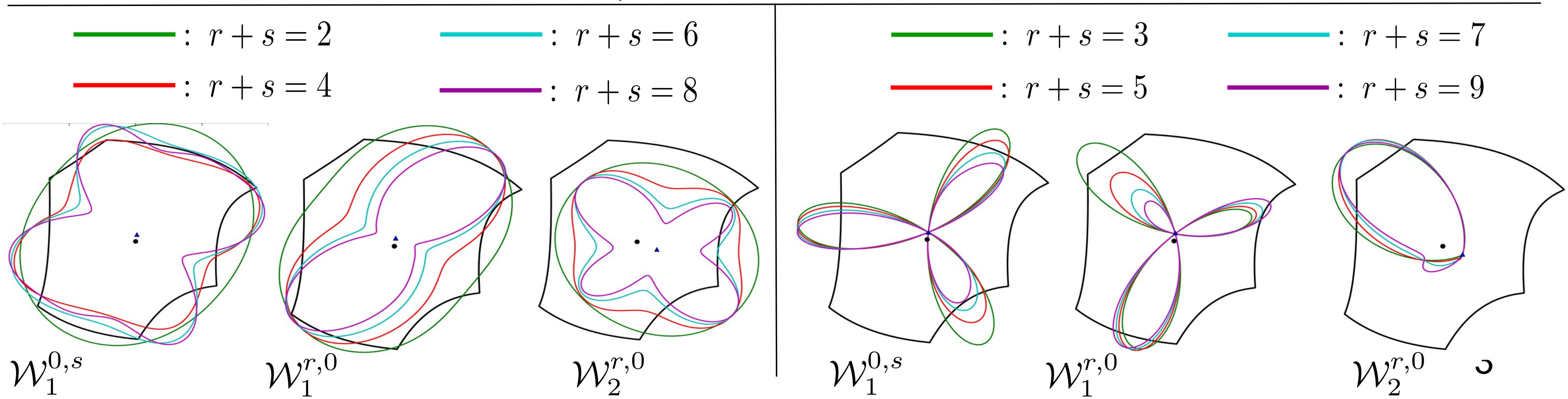
<u>Measures of surface distribution:</u>

$$\mathcal{W}_1^{r,s} = \int_{\partial\Omega_\alpha} \underline{x}^{\otimes^r} \odot [\underline{n}(\underline{x})]^{\otimes^s} \mathrm{d}S$$

<u>Curvature-weighted measures of surface distribution:</u>

$$\mathcal{W}_2^{r,s} = \int_{\partial\Omega_\alpha} \kappa(\underline{x})\underline{x}^{\otimes^r} \odot [\underline{n}(\underline{x})]^{\otimes^s} \mathrm{d}S$$

Reynolds glyphs of Minkowski tensors



Lippmann-Schwinger equation for periodic elastic media

Periodic elastic BVP:

$$\boldsymbol{\sigma}(\underline{x}) = \mathbb{L}(\underline{x}) : \boldsymbol{\varepsilon}(\underline{x}) \,, \qquad \nabla \cdot \boldsymbol{\sigma}(\underline{x}) = \underline{0} \,, \qquad \boldsymbol{\varepsilon}(\underline{x}) = \{\nabla \underline{u}(\underline{x})\}_{sym}$$

for all $\underline{x} \in \mathbb{R}^2$, with $\mathbb{L}(\underline{x} + (n\underline{e}_1 + m\underline{e}_2)L) = \mathbb{L}(\underline{x})$ for all $n, m \in \mathbb{Z}$ s.t.

$$\underline{u}(\underline{x} + (n\underline{e}_1 + m\underline{e}_2)L) = \underline{u}(\underline{x}) + L\,\overline{\boldsymbol{\varepsilon}} \cdot (n\underline{e}_1 + m\underline{e}_2)$$

$$\boldsymbol{\sigma}(\underline{x} + (n\underline{e}_1 + m\underline{e}_2)) \cdot \underline{e}_k = \boldsymbol{\sigma}(\underline{x}) \cdot \underline{e}_k \text{ for } k = 1, 2$$

and where $\overline{\bullet} := \dfrac{1}{L^2}\displaystyle\int_\Omega \bullet(\underline{x})\mathrm{d}\nu_{\underline{x}}$ is a volume average over $\Omega := [0, L] \times [0, L]$.

Then, as we introduce the polarization field $\boldsymbol{\tau}$ with reference $\mathbb{L}^0$,

$$\boldsymbol{\tau}(\underline{x}) := \boldsymbol{\sigma}(\underline{x}) - \mathbb{L}^0 : \boldsymbol{\varepsilon}(\underline{x}) = \Delta\mathbb{L}(\underline{x}) : \boldsymbol{\varepsilon}(\underline{x})$$

where $\Delta\mathbb{L}(\underline{x}) := \mathbb{L}(\underline{x}) - \mathbb{L}^0$, the local statement of equilibrium becomes

$$\nabla \cdot \boldsymbol{\tau}(\underline{x}) + \nabla \cdot [\mathbb{L}^0 : \boldsymbol{\varepsilon}(\underline{x})] = \underline{0}$$

Disturbance strain field $\tilde{\varepsilon}(\underline{x})$ with vanishing field average.

with solution

$$\boldsymbol{\varepsilon}(\underline{x}) = \overline{\boldsymbol{\varepsilon}} - \boxed{\boldsymbol{\Gamma} * \boldsymbol{\tau}(\underline{x})} = \overline{\boldsymbol{\varepsilon}} - \boldsymbol{\Gamma} * [\Delta\mathbb{L} : \boldsymbol{\varepsilon}(\underline{x})]$$

Lippmann-Schwinger equation

in which $\boldsymbol{\Gamma} * \boldsymbol{\tau}(\underline{x}) := \displaystyle\int_{\mathbb{R}^2} \boldsymbol{\Gamma}(\underline{x}' - \underline{x}) : \boldsymbol{\tau}(\underline{x}')\,\mathrm{d}\nu_{\underline{x}'}$.

Periodic Green operator for strains.

Note that for all $\underline{x}$, we have $\overline{\boldsymbol{\varepsilon}} = [\Delta\mathbb{L}(\underline{x})]^{-1} : \boldsymbol{\tau}(\underline{x}) + \boldsymbol{\Gamma} * \boldsymbol{\tau}(\underline{x})$

Hashin-Shtrikman (HS) variational principle

Multiplying the previous expression by a test field $\boldsymbol{\tau}'$, we have

$$\boldsymbol{\tau}'(\underline{x}) : \overline{\boldsymbol{\varepsilon}} = \boldsymbol{\tau}'(\underline{x}) : [\Delta\mathbb{L}(\underline{x})]^{-1} : \boldsymbol{\tau}(\underline{x}) + \boldsymbol{\tau}'(\underline{x}) : (\boldsymbol{\Gamma} * \boldsymbol{\tau})(\underline{x})$$

which, after volume averaging over Ω, becomes

$$\boxed{\overline{\boldsymbol{\tau}'} : \overline{\boldsymbol{\varepsilon}} = \overline{\boldsymbol{\tau}' : \Delta\mathbb{L}^{-1} : \boldsymbol{\tau}} + \overline{\boldsymbol{\tau}' : (\boldsymbol{\Gamma} * \boldsymbol{\tau})}}$$

Differential of the HS functional evaluated at the equilibrated stress $\boldsymbol{\tau}$

The HS functional is defined as follows by Hashin and Shtrikman (1962):

$$\boxed{\mathcal{H}(\boldsymbol{\tau}') := \overline{\boldsymbol{\tau}'} : \overline{\boldsymbol{\varepsilon}} - 1/2\,\overline{\boldsymbol{\tau}' : (\Delta\mathbb{L})^{-1} : \boldsymbol{\tau}'} - 1/2\,\overline{\boldsymbol{\tau}' : (\boldsymbol{\Gamma} * \boldsymbol{\tau}')}}$$

$\mathcal{H}$ admits a *stationary state* for the equilibrated polarization field $\boldsymbol{\tau}$, *irrespectively of the reference stiffness* $\mathbb{L}^0$. *At equilibrium*, we also have $\mathcal{H}(\boldsymbol{\tau}) = 1/2\overline{\boldsymbol{\varepsilon}} : (\mathbb{L}^{eff} - \mathbb{L}^0) : \overline{\boldsymbol{\varepsilon}}$, where $\mathbb{L}^{eff}$ is s.t. $\overline{\boldsymbol{\sigma}} = \mathbb{L}^{eff} : \overline{\boldsymbol{\varepsilon}}$.

Boundedness conditions of $\mathcal{H}$:

$$\Delta\mathbb{L}(\underline{x}) \text{ PSD for all } \underline{x} \text{ implies } \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathcal{V} \implies \sup_{\mathcal{V}_1}\mathcal{H} \leq \sup_{\mathcal{V}_2}\mathcal{H} \leq \sup_{\mathcal{V}}\mathcal{H} = \mathcal{H}(\boldsymbol{\tau})$$

$$\Delta\mathbb{L}(\underline{x}) \text{ NSD for all } \underline{x} \text{ implies } \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathcal{V} \implies \inf_{\mathcal{V}_1}\mathcal{H} \geq \inf_{\mathcal{V}_2}\mathcal{H} \geq \inf_{\mathcal{V}}\mathcal{H} = \mathcal{H}(\boldsymbol{\tau})$$

Searching for polarization fields among richer functional spaces guarantees not to deteriorate the quality of the solution if the reference medium is chosen properly.

Case of piecewise constant polarization fields, i.e. $\mathcal{V}^{h_0}$

Assume $\boldsymbol{\tau}^{h_0}(\underline{x}) := \sum_\alpha \chi_\alpha(\underline{x})\boldsymbol{\tau}^{(\alpha)}$ where $\chi_\alpha := \begin{cases} 1 & \text{if } \underline{x} \in \Omega_\alpha \\ 0 & \text{otherwise} \end{cases}$.

Then $\overline{\boldsymbol{\tau}^{h_0} : (\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_0})} = \sum_\alpha \sum_\gamma \boldsymbol{\tau}^\alpha : \mathbb{T}^{\alpha\gamma}_{0,0} : \boldsymbol{\tau}^\gamma$, where

influence tensors

$$\boxed{\mathbb{T}^{\alpha\gamma}_{0,0} := \frac{1}{|\Omega|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi_\alpha(\underline{x})\chi_\gamma(\underline{y})\boldsymbol{\Gamma}(\underline{x}-\underline{y})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}}$$

so that the HS functional becomes

$$\mathcal{H}(\boldsymbol{\tau}) = \sum_\alpha c_\alpha \boldsymbol{\tau}^\alpha : \overline{\boldsymbol{\varepsilon}} - \frac{1}{2}\sum_\alpha c_\alpha \boldsymbol{\tau}^\alpha : (\Delta\mathbb{L}^\alpha)^{-1} : \boldsymbol{\tau}^\alpha - \frac{1}{2}\sum_\alpha\sum_\gamma \boldsymbol{\tau}^\alpha : \mathbb{T}^{\alpha\gamma}_{0,0} : \boldsymbol{\tau}^\gamma$$

for which the stationary state $\hat{\boldsymbol{\tau}}^h(\underline{x}) = \inf_{\boldsymbol{\tau}^h(\underline{x})\in\mathcal{V}^h} \mathcal{H}(\boldsymbol{\tau}^h)$ is such that

$$c_\alpha(\Delta\mathbb{L}^\alpha)^{-1} : \boldsymbol{\tau}^\alpha + \sum_\gamma \mathbb{T}^{\alpha\gamma}_{0,0} : \boldsymbol{\tau}^\gamma = c_\alpha\overline{\boldsymbol{\varepsilon}} \quad \text{for all } \alpha$$

Remark: We want to avoid integrating $\boldsymbol{\Gamma}$. Instead, we want to find a relation between $\mathbb{T}^{\alpha\gamma}_{0,0}$, the Minkowski tensors (which we use to characterize morphological anisotropy) of the microstructure, and the derivatives of $\boldsymbol{\Gamma}$.

Taylor expansion of Green operators (1/2)

To avoid singularities, we introduce $\chi'_\alpha : \underline{x} \mapsto \chi_\alpha(\underline{x} + \underline{x}_\alpha)$ and

$\Omega'_\alpha := \{\underline{x} - \underline{x}_\alpha \mid \underline{x} \in \Omega_\alpha\}$ for all α so that

$$\mathbb{T}^{\alpha\gamma}_{0,0} = \frac{1}{|\Omega|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi'_\alpha(\underline{x})\chi'_\gamma(\underline{y})\mathbf{\Gamma}(\underline{x} - \underline{y} + \underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}} = \frac{1}{|\Omega|}\int_{\Omega'_\gamma}\int_{\Omega'_\alpha} \mathbf{\Gamma}(\underline{x} - \underline{y} + \underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

where $\underline{x}_{\gamma\alpha} := \underline{x}_\alpha - \underline{x}_\gamma$. Then for some basis $\{\underline{e_i}\}_{i=1,\dots,d}$ we have

$$\Gamma_{ijkl}(\underline{x} - \underline{y} + \underline{x}_{\gamma\alpha}) = \Gamma_{ijkl}(\underline{x}_{\gamma\alpha} - \underline{y}) + \Gamma_{ijkl,m}(\underline{x}_{\gamma\alpha} - \underline{y})x_m + (1/2!)\Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha} - \underline{y})x_m x_n + (1/3!)\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha} - \underline{y})x_m x_n x_o + \dots$$

and, similarly

$$\Gamma_{ijkl}(\underline{x}_{\gamma\alpha} - \underline{y}) = \Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,m}(\underline{x}_{\gamma\alpha})y_m + (1/2!)\Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha})y_m y_n - (1/3!)\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha})y_m y_n y_o + \dots$$

$$\Gamma_{ijkl,m}(\underline{x}_{\gamma\alpha} - \underline{y}) = \Gamma_{ijkl,m}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha})y_n + (1/2!)\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha})y_n y_o - (1/3!)\Gamma_{ijkl,mnop}(\underline{x}_{\gamma\alpha})y_n y_o y_p + \dots$$

$$\Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha} - \underline{y}) = \Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha})y_o + (1/2!)\Gamma_{ijkl,mnop}(\underline{x}_{\gamma\alpha})y_o y_p - (1/3!)\Gamma_{ijkl,mnopq}(\underline{x}_{\gamma\alpha})y_o y_p y_q + \dots$$

$$\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha} - \underline{y}) = \Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,mnop}(\underline{x}_{\gamma\alpha})y_p + (1/2!)\Gamma_{ijkl,mnopq}(\underline{x}_{\gamma\alpha})y_p y_q - (1/3!)\Gamma_{ijkl,mnopqr}(\underline{x}_{\gamma\alpha})y_p y_q y_r + \dots$$

Taylor expansion of Green operators (2/2)

Then we have

$$\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = \Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,m}(\underline{x}_{\gamma\alpha})y_m + (1/2!)\Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha})y_m y_n - (1/3!)\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha})y_m y_n y_o + \ldots$$
$$+ \left[\Gamma_{ijkl,m}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha})y_n + (1/2!)\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha})y_n y_o - (1/3!)\Gamma_{ijkl,mnop}(\underline{x}_{\gamma\alpha})y_n y_o y_p + \ldots\right] x_m$$
$$+ \frac{1}{2!}\left[\Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha})y_o + (1/2!)\Gamma_{ijkl,mnop}(\underline{x}_{\gamma\alpha})y_o y_p - (1/3!)\Gamma_{ijkl,mnopq}(\underline{x}_{\gamma\alpha})y_o y_p y_q + \ldots\right] x_m x_n$$
$$+ \frac{1}{3!}\left[\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha}) - \Gamma_{ijkl,mnop}(\underline{x}_{\gamma\alpha})y_p + (1/2!)\Gamma_{ijkl,mnopq}(\underline{x}_{\gamma\alpha})y_p y_q - (1/3!)\Gamma_{ijkl,mnopqr}(\underline{x}_{\gamma\alpha})y_p y_q y_r + \ldots\right] x_m x_n x_o$$
$$+ \ldots$$

that we recast in

$$\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = \Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) + \Gamma_{ijkl,m}(\underline{x}_{\gamma\alpha})[x_m - y_m]$$
$$+\Gamma_{ijkl,mn}(\underline{x}_{\gamma\alpha})\left[\frac{x_m x_n}{2!0!} - \frac{x_m y_n}{1!1!} + \frac{y_m y_n}{0!2!}\right]$$
$$+\Gamma_{ijkl,mno}(\underline{x}_{\gamma\alpha})\left[\frac{x_m x_n x_o}{3!0!} - \frac{x_m x_n y_o}{2!1!} + \frac{x_m y_n y_o}{1!2!} - \frac{y_m y_n y_o}{0!3!}\right]$$
$$+\Gamma_{ijkl,mnop}(\underline{x}_{\gamma\alpha})\left[\frac{x_m x_n x_o x_p}{4!0!} - \frac{x_m x_n x_o y_p}{3!1!} + \frac{x_m x_n y_o y_p}{2!2!} - \frac{x_m y_n y_o y_p}{1!3!} + \frac{y_m y_n y_o y_p}{0!4!}\right]$$
$$+\Gamma_{ijkl,mnopq}(\underline{x}_{\gamma\alpha})\left[\frac{x_m x_n x_o x_p x_q}{5!0!} - \frac{x_m x_n x_o x_p y_q}{4!1!} + \frac{x_m x_n x_o y_p y_q}{3!2!} - \frac{x_m x_n y_o y_p y_q}{2!3!} + \frac{x_m y_n y_o y_p y_q}{1!4!} - \frac{y_m y_n y_o y_p y_q}{0!5!}\right]$$
$$+ \ldots$$

Influence tensors for polarization fields in $\mathcal{V}^{h_0}$

Eventually, we obtain the following *n*-th order expansion

$$^{n}\boldsymbol{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) := \boldsymbol{\Gamma}(\underline{x}_{\gamma\alpha}) + \sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\left\langle \boldsymbol{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}), \underline{x}^{\otimes^{k-i}} \otimes \underline{y}^{\otimes^{i}} \right\rangle_k \text{ for all } (\underline{x},\underline{y}) \in \Omega'_\alpha \times \Omega'_\gamma$$

where $\Omega_\alpha \cap \Omega_\gamma = \emptyset$, which we use to construct the following estimate of influence tensors for $\alpha \neq \gamma$:

$$^{n}\mathbb{T}^{\alpha\gamma}_{0,0} := \frac{1}{|\Omega|}\int_{\Omega'_\gamma}\int_{\Omega'_\alpha} {}^{n}\boldsymbol{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

$$^{n}\mathbb{T}^{\alpha\gamma}_{0,0} = c_\alpha c_\gamma \boldsymbol{\Gamma}(\underline{x}_{\gamma\alpha}) + \frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\left\langle \boldsymbol{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}), \mathcal{W}_0^{k-i,0}(\Omega'_\alpha) \otimes \mathcal{W}_0^{i,0}(\Omega'_\gamma) \right\rangle_k$$

where, $\boldsymbol{\Gamma}^{(m)}(\underline{x})$ is the *m*-th derivative of the Green operator, i.e. with components $\Gamma^{(m)}_{ijkln_1\dots n_m}(\underline{x}) = \partial_{n_1\dots n_m}\Gamma_{ijkl}(\underline{x})$,

$\langle \bullet, \bullet \rangle_k$ are “appropriate inner products” for $k \geq 1$

- Maxwell-Betti theorem $\implies \Gamma_{ijkl}(\underline{x},\underline{y}) = \Gamma_{klij}(\underline{y},\underline{x})$

  Then, stationarity $\implies \Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = \Gamma_{klij}(\underline{y}-\underline{x}+\underline{x}_{\alpha\gamma})$

  However, we don’t know if $^{n}\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = {}^{n}\Gamma_{klij}(\underline{y}-\underline{x}+\underline{x}_{\alpha\gamma})$ is true.
- To verify, we define a symmetrized expansion, …

Computing components of $\left\langle \mathbf{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}), \underline{x}^{\otimes^{k-i}} \otimes \underline{y}^{\otimes^{i}} \right\rangle_k$

- The component $\Gamma^{(k)}_{ijkl,k_1\dots k_k}(\underline{x}_{\gamma\alpha}) x_{k_1\dots k_{k-i}} y_{k_{k-i+1}\dots k_k}$ consists of the sum of $(k-i+1)(i+1)$ possibly different terms of the form

$$ {}^{k,i}A_{ijkl}(n_1^\alpha, n_1^\gamma) := \Gamma^{(k)}_{ijkl,\underbrace{11\dots1}_{(n_1^\alpha+n_1^\gamma \text{ times})}\underbrace{22\dots2}_{(k-n_1^\alpha-n_1^\gamma \text{ times})}}(\underline{x}_{\gamma\alpha})\, x_1^{n_1^\alpha} x_2^{(k-i)-n_1^\alpha} y_1^{n_1^\gamma} y_2^{i-n_1^\gamma} $$

  where $n_1^\alpha \in [0, k-i]$ and $n_1^\gamma \in [0, i]$. To account for the repetition of combinations of indices, we have

$$ \Gamma^{(k)}_{ijkl,k_1\dots k_k}(\underline{x}_{\gamma\alpha}) x_{k_1\dots k_{k-i}} y_{k_{k-i+1}\dots k_k} = \sum_{n_1^\alpha=0}^{k-i} \sum_{n_1^\gamma=0}^{i} \binom{k-i+1}{n_1^\alpha} \binom{i+1}{n_1^\gamma} {}^{k,i}A_{ijkl}(n_1^\alpha, n_1^\gamma) \quad . $$

- We recall that the component $\Gamma^{(k)}_{ijkl,\underbrace{11\dots1}_{(n_1^\alpha+n_1^\gamma \text{ times})}\underbrace{22\dots2}_{(k-n_1^\alpha-n_1^\gamma \text{ times})}}(\underline{x}_{\gamma\alpha})$ of the gradients of the Green operator for strain are stored in $\mathrm{d}\Gamma[\alpha][\gamma-\alpha-1][i_{ijkl}][k][n_1^\alpha+n_1^\gamma]$ if $\alpha < \gamma$, or as $(-1)^k \mathrm{d}\Gamma[\gamma][\alpha-\gamma-1][i_{ijkl}][k][n_1^\alpha+n_1^\gamma]$ if $\alpha > \gamma$.

- Then,

Influence tensors for polarization fields in $\mathcal{V}^{h_0}$

We define the following symmetrized Taylor expansion:

$$
\begin{aligned}
{}^n\tilde{\Gamma}_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) &:= 1/2[{}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) + {}^n\Gamma_{klij}(\underline{y}-\underline{x}+\underline{x}_{\alpha\gamma})] \\
&= 1/2\left[\Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) + \Gamma_{klij}(\underline{x}_{\alpha\gamma})\right] \\
&+\frac{1}{2}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\Gamma^{(k)}_{ijklk_1\ldots k_k}(\underline{x}_{\gamma\alpha})x_{k_1}\ldots x_{k_{k-i}}y_{k_{k-i+1}}\ldots y_{k_k} \\
&+\frac{1}{2}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i(-1)^k}{(k-i)!i!}\Gamma^{(k)}_{klijk_1\ldots k_k}(\underline{x}_{\alpha\gamma})x_{k_1}\ldots x_{k_{k-i}}y_{k_{k-i+1}}\ldots y_{k_k}
\end{aligned}
$$

where $\mathbf{\Gamma}(\underline{x}) = \mathbf{\Gamma}(-\underline{x})$ and $\Gamma_{ijkl}(\underline{x}) = \Gamma_{klij}(\underline{x}) \implies \Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) = \Gamma_{klij}(\underline{x}_{\alpha\gamma})$,

which implies $\Gamma^{(k)}_{ijklk_1\ldots k_k}(\underline{x}_{\gamma\alpha}) = (-1)^k\Gamma^{(k)}_{klijk_1\ldots k_k}(\underline{x}_{\alpha\gamma})$

so that we have $\qquad = (-1)^k\Gamma^{(k)}_{ijklk_1\ldots k_k}(\underline{x}_{\alpha\gamma})$

$$
\begin{aligned}
{}^n\tilde{\Gamma}_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) &= \Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) \\
&+\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})}{2(k-i)!i!}\left[x_{k_1}\ldots x_{k_{k-i}}y_{k_{k-i+1}}\ldots y_{k_k} + \ldots\right. \\
&\left.\cdots + (-1)^{2k}x_{k_1}\ldots x_{k_{k-i}}y_{k_{k-i+1}}\ldots y_{k_k}\right]
\end{aligned}
$$

leading up to

$${}^n\tilde{\Gamma}_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = {}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})$$

$$\implies \boxed{{}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = {}^n\Gamma_{klij}(\underline{y}-\underline{x}+\underline{x}_{\alpha\gamma})}$$

Self-influence tensors for polarization fields in $\mathcal{V}^{h_0}$

When $\gamma = \alpha$, we refer to $\mathbb{T}^{\alpha\gamma}_{0,0}$ as a self-influence tensor. We then have

$$\mathbb{T}^{\alpha\alpha}_{0,0} := \frac{1}{|\Omega|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\chi_\alpha(\underline{x})\chi_\alpha(\underline{y})\mathbf{\Gamma}(\underline{x}-\underline{y})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

which we recast in

$$\mathbb{T}^{\alpha\alpha}_{0,0} = \frac{1}{|\Omega|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\chi_\alpha(\underline{x}+\underline{x}_\alpha)\chi_\alpha(\underline{y}+\underline{x}_\gamma)\mathbf{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

for some $\gamma \neq \alpha$ and where $\underline{x}_{\gamma\alpha} := \underline{x}_\alpha - \underline{x}_\gamma$, so that we obtain

$$\mathbb{T}^{\alpha\alpha}_{0,0} = \frac{1}{|\Omega|}\int_{\Omega^\gamma_\alpha}\int_{\Omega'_\alpha}\mathbf{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

where $\Omega^\gamma_\alpha := \Omega_\alpha \uplus \{-\underline{x}_\gamma\} = \{\underline{x} - \underline{x}_\gamma \mid \underline{x} \in \Omega_\alpha\}$. Using the same Taylor series expansion as before, we have

$$^n\mathbb{T}^{\alpha\alpha}_{0,0} = \frac{1}{|\Omega|}\int_{\Omega^\gamma_\alpha}\int_{\Omega'_\alpha}\mathbf{\Gamma}(\underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}} + \sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\left\langle \mathbf{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}), \frac{1}{|\Omega|}\int_{\Omega^\gamma_\alpha}\int_{\Omega'_\alpha}\underline{x}^{\otimes^{k-i}}\otimes\underline{y}^{\otimes^i}\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\right\rangle_k$$

which becomes

$$^n\mathbb{T}^{\alpha\alpha}_{0,0} = c^2_\alpha\mathbf{\Gamma}(\underline{x}_{\gamma\alpha}) + \frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\left\langle \mathbf{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}), \mathcal{W}^{k-i,0}_0(\Omega'_\alpha)\otimes\mathcal{W}^{i,0}_0(\Omega^\gamma_\alpha)\right\rangle_k$$

where we recall that $\mathcal{W}^{i,0}_0(\bullet)$ is motion covariant and that $\Omega^\gamma_\alpha = \Omega'_\alpha \uplus \{\underline{x}_{\gamma\alpha}\}$ so that, for *i>0*, we have

$$\boxed{\mathcal{W}^{i,0}_0(\Omega^\gamma_\alpha) = \sum_{t=0}^{i}\binom{i}{t}\underline{x}^{\otimes^t}_{\gamma\alpha}\odot\mathcal{W}^{i-t,0}_0(\Omega'_\alpha)}$$

Compute these for *i = 0,…,n*

Influence tensors for polarization fields in $\mathcal{V}^{h_0}$

To summarize, the following estimates of influence and self-influence tensors are obtained:

estimate of the 0-0 influence tensor of Ω_γ over Ω_α

$$ {}^n\mathbb{T}^{\alpha\gamma}_{0,0} := \frac{1}{|\Omega|}\int_{\Omega'_\gamma}\int_{\Omega'_\alpha} {}^n\mathbf{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}} $$

$\gamma \neq \alpha$

estimate of the 0-0 *self-influence tensor of* Ω_α

$$ {}^n\mathbb{T}^{\alpha\alpha}_{0,0} = \frac{1}{|\Omega|}\int_{\Omega^\gamma_\alpha}\int_{\Omega'_\alpha} {}^n\mathbf{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}} $$

$\gamma \neq \alpha$

which we respectively recast in the following expressions:

$$ ({}^nT^{\alpha\gamma}_{0,0})_{ijkl} = c_\alpha c_\gamma \Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) + \sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})}{(k-i)!i!|\Omega|}[W_0^{k-i,0}(\Omega'_\alpha)]_{k_1..k_{k-i}}[W_0^{i,0}(\Omega'_\gamma)]_{k_{k-i+1}..k_k} $$

for all $\gamma \neq \alpha$

$$ \left.\begin{array}{l} {}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = {}^n\Gamma_{klij}(\underline{y}-\underline{x}+\underline{x}_{\alpha\gamma}) \\ {}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) = {}^n\Gamma_{klij}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) \end{array}\right] \Longrightarrow \boxed{({}^nT^{\gamma\alpha}_{0,0})_{klij} = ({}^nT^{\gamma\alpha}_{0,0})_{ijkl} = ({}^nT^{\alpha\gamma}_{0,0})_{klij} = ({}^nT^{\alpha\gamma}_{0,0})_{ijkl}} $$

$$ ({}^nT^{\alpha\alpha}_{0,0})_{ijkl} = c_\alpha^2 \Gamma_{ijkl}(\underline{x}_{\gamma\alpha}) + \frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i\ {}^n\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})}{(k-i)!i!}[W_0^{k-i,0}(\Omega'_\alpha)]_{k_1..k_{k-i}}[W_0^{i,0}(\Omega^\gamma_\alpha)]_{k_{k-i+1}..k_k} $$

for any $\gamma \neq \alpha$

For γ fixed, $\boxed{({}^nT^{\alpha\alpha}_{0,0})_{ijkl} = ({}^nT^{\alpha\alpha}_{0,0})_{klij}}$

Piecewise polynomial polarization fields, i.e. $\mathcal{V}^{h_p}$

Assume a trial polynomial field of degree p given by

$$\boldsymbol{\tau}^{h_p}(\underline{x}) := \sum_\alpha \left(\chi_\alpha(\underline{x})\boldsymbol{\tau}^\alpha + \chi_\alpha(\underline{x}) \sum_{k=1}^{p} \left\langle \boldsymbol{\tau}^\alpha \boldsymbol{\partial}^k, (\underline{x} - \underline{x}^\alpha)^{\otimes k} \right\rangle_k \right)$$

where $(\tau^\alpha \partial^k)_{ijk_1,\dots,k_k} := \partial^k(\tau^\alpha_{ij})/\partial x_{k_1}\dots\partial x_{k_k}$

$(\partial^k \tau^\alpha)_{k_1,\dots,k_k ij} := \partial^k(\tau^\alpha_{ij})/\partial x_{k_1}\dots\partial x_{k_k}$ $\quad \partial^k_{k_1\dots k_k}\tau^\alpha_{ij}$

so that we have

$$\overline{\boldsymbol{\tau}^{h_p} : (\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_p})} =$$

$$\sum_\alpha \sum_\gamma \tau^\alpha_{ij} \int_{\Omega_\alpha}\int_{\Omega_\gamma} \Gamma_{ijkl}(\underline{x}-\underline{y})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\tau^\gamma_{kl} + \tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}\Gamma_{ijkl}(\underline{x}-\underline{y})(y_r - x^\gamma_r)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\partial_r\tau^\gamma_{kl}$$

$(T^{\alpha\gamma}_{0,0})_{ijkl}$ $\quad (T^{\alpha\gamma}_{0,1})_{ijklr}$

$$+\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}\Gamma_{ijkl}(\underline{x}-\underline{y})(y_r - x^\gamma_r)(y_s - x^\gamma_s)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\partial^2_{rs}\tau^\gamma_{kl} + \dots$$

$(T^{\alpha\gamma}_{0,2})_{ijklrs}$

$$+\partial_r\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_r - x^\alpha_r)\Gamma_{ijkl}(\underline{x}-\underline{y})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\tau^\gamma_{kl} + \partial_r\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_r - x^\alpha_r)\Gamma_{ijkl}(\underline{x}-\underline{y})(y_s - x^\gamma_s)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\partial_s\tau^\gamma_{kl}$$

$(T^{\alpha\gamma}_{1,0})_{rijkl}$ $\quad (T^{\alpha\gamma}_{1,1})_{rijkls}$

$$+\partial_r\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_r - x^\alpha_r)\Gamma_{ijkl}(\underline{x}-\underline{y})(y_s - x^\gamma_s)(y_t - x^\gamma_t)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\partial^2_{st}\tau^\gamma_{kl} + \dots$$

$(T^{\alpha\gamma}_{1,2})_{rijklst}$

$$+\partial^2_{rs}\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_r - x^\alpha_r)(x_s - x^\alpha_s)\Gamma_{ijkl}(\underline{x}-\underline{y})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\tau^\gamma_{kl}$$

$(T^{\alpha\gamma}_{2,0})_{rsijkl}$

$$+\partial^2_{rs}\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_r - x^\alpha_r)(x_s - x^\alpha_s)\Gamma_{ijkl}(\underline{x}-\underline{y})(y_t - x^\gamma_t)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\partial_s\tau^\gamma_{kl}$$

$(T^{\alpha\gamma}_{2,1})_{rsijklt}$

$$+\partial^2_{rs}\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_r - x^\alpha_r)(x_s - x^\alpha_s)\Gamma_{ijkl}(\underline{x}-\underline{y})(y_t - x^\gamma_t)(y_u - x^\gamma_u)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\partial^2_{tu}\tau^\gamma_{kl} + \dots$$

$(T^{\alpha\gamma}_{2,2})_{rsijkltu}$

Let's look at this term for $r, s \le p$

$(T^{\alpha\gamma}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}$

$$+\dots$$

$$+\partial^p_{r_1\dots r_r}\tau^\alpha_{ij}\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_{r_1} - x^\alpha_{r_1})\dots(x_{r_r} - x^\alpha_{r_r})\Gamma_{ijkl}(\underline{x}-\underline{y})(y_{s_1} - x^\gamma_{s_1})\dots(y_{s_s} - x^\gamma_{s_s})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}\partial^q_{s_1\dots s_s}\tau^\gamma_{kl} + \dots$$

Influence tensors for polarization fields in $\mathcal{V}^{h_p}$

From the previous expression, we want to address the terms with components of the form

$$\int_{\Omega_\alpha}\int_{\Omega_\gamma}(x_{r_1}-x^\alpha_{r_1})\dots(x_{r_r}-x^\alpha_{r_r})\Gamma_{ijkl}(\underline{x}-\underline{y})(y_{s_1}-x^\gamma_{s_1})\dots(y_{s_s}-x^\gamma_{s_s})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

$$=$$

Change of variable

$$\int_{\Omega'_\alpha}\int_{\Omega'_\gamma}x_{r_1}\dots x_{r_r}\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})y_{s_1}\dots y_{s_s}\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$ with $\Omega'_\bullet := \{\underline{x}-\underline{x}_\bullet \mid \underline{x}\in\Omega_\bullet\}$,

where we used the same change of variables as previously. Now, from

$${}^n\boldsymbol{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})=\boldsymbol{\Gamma}(\underline{x}_{\gamma\alpha})+\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\left\langle\boldsymbol{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}),\underline{x}^{\otimes^{k-i}}\otimes\underline{y}^{\otimes^{i}}\right\rangle_k \text{ for all } (\underline{x},\underline{y})\in\Omega'_\alpha\times\Omega'_\gamma$$

we obtain the following estimate of "<u>*r-s* influence tensor of</u> Ω_γ <u>over</u> Ω_α"

$$({}^nT^{\alpha\gamma}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}:=\frac{1}{|\Omega|}\int_{\Omega'_\gamma}\int_{\Omega'_\alpha}x_{r_1}\dots x_{r_r}\,{}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})y_{s_1}\dots y_{s_s}\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

defined for $r,s\le p$ and which we recast as follows:

$$({}^nT^{\alpha\gamma}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}=\frac{1}{|\Omega|}[W^{r,0}_0(\Omega'_\alpha)]_{r_1\dots r_r}\Gamma_{ijkl}(\underline{x}_{\gamma\alpha})[W^{s,0}_0(\Omega'_\gamma)]_{s_1\dots s_s}$$
$$+\frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})[W^{r+k-i,0}_0(\Omega'_\alpha)]_{k_1..k_{k-i}r_1..r_r}[W^{i+s,0}_0(\Omega'_\gamma)]_{k_{k-i+1}..k_k s_1..s_s}$$

$$\left.\begin{aligned}{}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})&={}^n\Gamma_{klij}(\underline{y}-\underline{x}+\underline{x}_{\alpha\gamma})\\{}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})&={}^n\Gamma_{klij}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})\end{aligned}\right]\begin{aligned}&\Longrightarrow({}^nT^{\alpha\gamma}_{r,s})_{r_1\dots r_r klij s_1\dots s_s}=({}^nT^{\alpha\gamma}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}\\&\Longrightarrow({}^nT^{\gamma\alpha}_{s,r})_{s_1\dots s_s klij r_1\dots r_r}=({}^nT^{\alpha\gamma}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}\end{aligned}$$

Self-influence tensors for polarization fields in $\mathcal{V}^{h_p}$

Similarly as before, we want to address the terms with those components:

$$\int_{\Omega_\alpha}\int_{\Omega_\alpha}(x_{r_1}-x^\alpha_{r_1})\dots(x_{r_r}-x^\alpha_{r_r})\Gamma_{ijkl}(\underline{x}-\underline{y})(y_{s_1}-x^\alpha_{s_1})\dots(y_{s_s}-x^\alpha_{s_s})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

$$=$$ ↓ Change of variable

$$\int_{\Omega^\gamma_\alpha}\int_{\Omega'_\alpha}x_{r_1}\dots x_{r_r}\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})(y_{s_1}-x^{\gamma\alpha}_{s_1})\dots(y_{s_s}-x^{\gamma\alpha}_{s_s})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

$\Omega'_\alpha := \Omega_\alpha \uplus \{-\underline{x}_\alpha\}$

$\Omega^\gamma_\alpha := \Omega_\alpha \uplus \{-\underline{x}_\gamma\}$

$= \Omega'_\alpha \uplus \{\underline{x}_{\gamma\alpha}\}$

where, again, we have $^n\mathbf{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})=\mathbf{\Gamma}(\underline{x}_{\gamma\alpha})+\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\left\langle\mathbf{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}),\underline{x}^{\otimes^{k-i}}\otimes\underline{y}^{\otimes^i}\right\rangle_k$

so that an estimate of the "<u>*r-s* self-influence tensor of</u> Ω_α" is obtained by

$$({}^nT^{\alpha\alpha}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}:=\frac{1}{|\Omega|}\int_{\Omega^\gamma_\alpha}\int_{\Omega'_\alpha}x_{r_1}\dots x_{r_r}\,{}^n\Gamma_{ijkl}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})(y_{s_1}-x^{\gamma\alpha}_{s_1})\dots(y_{s_s}-x^{\gamma\alpha}_{s_s})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}$$

which we recast in

$r,s\le p$

$$({}^nT^{\alpha\alpha}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}=\frac{1}{|\Omega|}[W^{r,0}_0(\Omega'_\alpha)]_{r_1\dots r_r}\Gamma_{ijkl}(\underline{x}_{\gamma\alpha})\int_{\Omega^\gamma_\alpha}(y_{s_1}-x^{\gamma\alpha}_{s_1})\dots(y_{s_s}-x^{\gamma\alpha}_{s_s})\mathrm{d}\nu_{\underline{y}}$$

$$+\frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})[W^{r+k-i,0}_0(\Omega'_\alpha)]_{k_1..k_{k-i}r_1..r_r}\int_{\Omega^\gamma_\alpha}y_{k-i+1}\dots y_k(y_{s_1}-x^{\gamma\alpha}_{s_1})\dots(y_{s_s}-x^{\gamma\alpha}_{s_s})\mathrm{d}\nu_{\underline{y}}$$

and in $({}^nT^{\alpha\alpha}_{r,s})_{r_1\dots r_r ijkl s_1\dots s_s}=\frac{1}{|\Omega|}[W^{r,0}_0(\Omega'_\alpha)]_{r_1\dots r_r}\Gamma_{ijkl}(\underline{x}_{\gamma\alpha})\int_{\Omega'_\alpha}y_{s_1}\dots y_{s_s}\mathrm{d}\nu_{\underline{y}}$

$$+\frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})[W^{r+k-i,0}_0(\Omega'_\alpha)]_{k_1..k_{k-i}r_1..r_r}\int_{\Omega'_\alpha}(y_{k-i+1}+x^{\gamma\alpha}_{k-i+1})\dots(y_k+x^{\gamma\alpha}_k)y_{s_1}\dots y_{s_s}\mathrm{d}\nu_{\underline{y}}$$

Self-influence tensors for polarization fields in $\mathcal{V}^{h_p}$

... so that $({}^{n}T^{\alpha\alpha}_{r,s})_{r_1\ldots r_r ijkl s_1\ldots s_s} = \frac{1}{|\Omega|}[W_0^{r,0}(\Omega'_\alpha)]_{r_1\ldots r_r}\Gamma_{ijkl}(\underline{x}_{\gamma\alpha})[W_0^{s,0}(\Omega'_\alpha)]_{s_1\ldots s_s}$

$$+\frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})[W_0^{r+k-i,0}(\Omega'_\alpha)]_{k_1..k_{k-i}r_1..r_r}\int_{\Omega'_\alpha}(y_{k-i+1}+x^{\gamma\alpha}_{k-i+1})\ldots(y_k+x^{\gamma\alpha}_k)y_{s_1}\ldots y_{s_s}\mathrm{d}\nu_{\underline{y}}$$

Note that $\int_{\Omega'_\alpha}(y_{k-i+1}+x^{\gamma\alpha}_{k-i+1})\ldots(y_k+x^{\gamma\alpha}_k)y_{s_1}\ldots y_{s_s}\mathrm{d}\nu_{\underline{y}}$ refers to the components of

$$\int_{\Omega'_\alpha}(\underline{y}+\underline{x}_{\gamma\alpha})^{\otimes^i}\otimes\underline{y}^{\otimes^s}\mathrm{d}\nu_{\underline{y}} = \int_{\Omega'_\alpha}\left[\sum_{t=0}^{i}\binom{i}{t}(\underline{x}_{\gamma\alpha})^{\otimes^{i-t}}\odot\underline{y}^{\otimes t}\right]\otimes\underline{y}^{\otimes s}\mathrm{d}\nu_{\underline{y}}$$

$$= \sum_{t=0}^{i}\binom{i}{t}(\underline{x}_{\gamma\alpha})^{\otimes^{i-t}}\overset{i-t,t}{\odot}\mathcal{W}_0^{t+s,0}(\Omega'_\alpha) =: {}^{\gamma}\widetilde{\mathcal{W}}_0^{i|s,0}(\Omega'_\alpha)$$

Requires to know
$\mathcal{W}_0^{s,0}(\Omega'_\alpha),\ \ldots,\ \mathcal{W}_0^{i+s,0}(\Omega'_\alpha)$

Eventually, we obtain the following estimates of the "<u>*r-s* self-influence tensor of Ω_α</u>":

$r, s \leq p$

$$({}^{n}T^{\alpha\alpha}_{r,s})_{r_1\ldots r_r ijkl s_1\ldots s_s} = \frac{1}{|\Omega|}[W_0^{r,0}(\Omega'_\alpha)]_{r_1\ldots r_r}\Gamma_{ijkl}(\underline{x}_{\gamma\alpha})[W_0^{s,0}(\Omega'_\alpha)]_{s_1\ldots s_s}$$

for any $\gamma \neq \alpha$

$$+\frac{1}{|\Omega|}\sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\Gamma^{(k)}_{ijklk_1..k_k}(\underline{x}_{\gamma\alpha})[W_0^{r+k-i,0}(\Omega'_\alpha)]_{k_1..k_{k-i}r_1..r_r}[{}^{\gamma}\widetilde{\mathcal{W}}_0^{i|s,0}(\Omega'_\alpha)]_{k_{k-i+1}\ldots k_k s_1\ldots s_s}$$

Most likely, $({}^{n}T^{\alpha\alpha}_{r,s})_{r_1\ldots r_r ijkl s_1\ldots s_s} \neq ({}^{n}T^{\alpha\alpha}_{s,r})_{s_1\ldots s_s ijkl r_1\ldots r_r}$. Consider having a symmetric estimate $({}^{n}\tilde{T}^{\alpha\alpha}_{r,s})_{r_1\ldots r_r ijkl s_1\ldots s_s}$.

What about local equilibrium of the polarization field?

- For a piecewise polynomial trial field given by

$$\boldsymbol{\tau}^{h_p}(\underline{x}) := \sum_{\alpha=0}^{n_\alpha-1}\left(\chi_\alpha(\underline{x})\boldsymbol{\tau}^\alpha + \chi_\alpha(\underline{x})\sum_{k=1}^{p}\left\langle \boldsymbol{\tau}^\alpha\boldsymbol{\partial}^k, (\Delta^\alpha\underline{x})^{\otimes^k}\right\rangle_k\right)$$

$$\tau_{ij}^{h_p}(\underline{x}) = \sum_{\alpha=0}^{n_\alpha-1}\chi_\alpha(\underline{x})\left[\tau_{ij}^\alpha + \sum_{k=1}^{p}\sum_{i=0}^{k}\binom{k}{n_1(i)}\tau_{ij}^\alpha\partial^k_{\underbrace{11\ldots1}_{(n_1(i)\text{ times})}\underbrace{22\ldots2}_{(k-n_1(i)\text{ times})}}(\Delta^\alpha x_1)^{n_1(i)}(\Delta^\alpha x_2)^{k-n_1(i)}\right]$$

where $\Delta^\alpha\underline{\underline{x}} := \underline{\underline{x}} - \underline{\underline{x}}^\alpha$.

```
tau(α,&tau0,&tau_grads,dx1,dx2):
tau=tau0[3α … 3(α+1)-1]
for k in [1… p]:
   istart=3nα((k-1)²+3(k-1))/2
   for i in [0… k]:
      if i%2==0: ni1=k-i/2     ; ni2=k-ni1
           else: ni2=k-(i-1)/2 ; ni1=k-ni2
      tau+=sqrt(Binom(k,ni1))*tau_grads[istart+3α(k+1)+3i… istart+3α(k+1)+3(i+1)-1]*dx1^ni1*dx2^(k-ni1)
tau[2]/=sqrt(2)
```

In Mandel representation: $\{\tau\}$, $\begin{bmatrix}\{\partial\tau\}\\ \{\partial^2\tau\}\\ \{\partial^3\tau\}\\ \vdots\\ \{\partial^p\tau\}\end{bmatrix}$

- A local error in equilibrium is given by $\epsilon(\underline{x}) := \|\nabla\cdot\boldsymbol{\tau}^{h_p}(\underline{x})\|\ \forall\ \underline{x}\in\Omega_\alpha$. We get

$$[\nabla\cdot\boldsymbol{\tau}^{h_p}(\underline{x})]\cdot\underline{e}_i = \tau_{ij}\partial_j^1 + \sum_{k=2}^{p} k\tau_{ij}\partial^k_{jk_1\ldots k_{k-1}}\Delta^\alpha x_{k_1}\ldots\Delta^\alpha x_{k_{k-1}}\ \forall\ \underline{x}\in\Omega'_\alpha$$

$$[\nabla\cdot\boldsymbol{\tau}^{h_p}(\underline{x})]\cdot\underline{e}_i = \tau_{ij}\partial_j^1 + \sum_{k=2}^{p}\sum_{i=0}^{k-1}\binom{k-1}{n_1(i)}k\tau_{ij}\partial^k_{j\underbrace{11\ldots1}_{(n_1(i))}\underbrace{22\ldots2}_{(k-1-n_1(i))}}(\Delta^\alpha x_1)^{n_1(i)}(\Delta^\alpha x_2)^{k-1-n_1(i)}$$

so that …

What about local equilibrium of the polarization field?

... we have the following components

$$[\nabla\cdot\boldsymbol{\tau}^{h_p}(\underline{x})]\cdot\underline{e}_1=\tau_{11}\partial_1^1+\sum_{k=2}^{p}\sum_{i=0}^{k-1}\binom{k-1}{n_1(i)}k\tau_{11}\partial^k\underbrace{\overbrace{11\ldots1}}_{(n_1(i)+1)}\underbrace{\overbrace{22\ldots2}}_{(k-1-n_1(i))}(\Delta^\alpha x_1)^{n_1(i)}(\Delta^\alpha x_2)^{k-1-n_1(i)}$$

$$+\tau_{12}\partial_2^1+\sum_{k=2}^{p}\sum_{i=0}^{k-1}\binom{k-1}{n_1(i)}k\tau_{12}\partial^k\underbrace{\overbrace{11\ldots1}}_{(n_1(i))}\underbrace{\overbrace{22\ldots2}}_{(k-n_1(i))}(\Delta^\alpha x_1)^{n_1(i)}(\Delta^\alpha x_2)^{k-1-n_1(i)}$$

and

$$[\nabla\cdot\boldsymbol{\tau}^{h_p}(\underline{x})]\cdot\underline{e}_2=\tau_{12}\partial_1^1+\sum_{k=2}^{p}\sum_{i=0}^{k-1}\binom{k-1}{n_1(i)}k\tau_{12}\partial^k\underbrace{\overbrace{11\ldots1}}_{(n_1(i)+1)}\underbrace{\overbrace{22\ldots2}}_{(k-1-n_1(i))}(\Delta^\alpha x_1)^{n_1(i)}(\Delta^\alpha x_2)^{k-1-n_1(i)}$$

$$+\tau_{22}\partial_2^1+\sum_{k=2}^{p}\sum_{i=0}^{k-1}\binom{k-1}{n_1(i)}k\tau_{22}\partial^k\underbrace{\overbrace{11\ldots1}}_{(n_1(i))}\underbrace{\overbrace{22\ldots2}}_{(k-n_1(i))}(\Delta^\alpha x_1)^{n_1(i)}(\Delta^\alpha x_2)^{k-1-n_1(i)}$$

and the following is implemented:

```
div_error(α,&tau0,&tau_grads,dx1,dx2):
  div_tau=[0,0]
  for k in [1… p]:
     istart=3nα((k-1)²+3(k-1))/2
     for i in [0… k]:
        if (i%2==0): ni1=(k-1)-i/2     ; ni2=k-ni1
               else: ni2=(k-1)-(i-1)/2 ; ni1=k-ni2
        if (ni1>0):
           fac=Binom(k-1,ni1-1)/sqrt(Binom(k,ni1))
           div_tau[0]+=fac*k*tau_grads[i_start+al*(k+1)*3+i*3]*dx1**(ni1-1)*dx2**ni2
           div_tau[1]+=fac*k*tau_grads[i_start+al*(k+1)*3+i*3+2]/sqrt(2)*dx1**(ni1-1)*dx2**ni2
        if (ni2>0):
           fac=Binom(k-1,ni2-1)/sqrt(Binom(k,ni1))
           div_tau[0]+=fac*k*tau_grads[i_start+al*(k+1)*3+i*3+2]/sqrt(2)*dx1**ni1*dx2**(ni2-1)
           div_tau[1]+=fac*k*tau_grads[i_start+al*(k+1)*3+i*3+1]*dx1**ni1*dx2**(ni2-1)
  return sqrt(div_tau[0]**2+div_tau[1]**2)
```

What about local equilibrium of the polarization field?

- For a piecewise polynomial trial field given by

$$\boldsymbol{\tau}^{h_p}(\underline{x}) := \sum_{\alpha} \left(\chi_\alpha(\underline{x})\boldsymbol{\tau}^\alpha + \chi_\alpha(\underline{x}) \sum_{k=1}^{p} \left\langle \boldsymbol{\tau}^\alpha \boldsymbol{\partial}^k, (\Delta^\alpha \underline{x})^{\otimes^k} \right\rangle_k \right)$$

where $\Delta^\alpha \underline{x} := \underline{x} - \underline{x}^\alpha$. Then,

$$\underline{0} = \nabla \cdot \boldsymbol{\tau}^{h_p}(\underline{x}) \; \forall \, \underline{x} \in \Omega'_\alpha \implies \underline{0} = \sum_{k=2}^{p} \left\langle \boldsymbol{\tau}^\alpha \boldsymbol{\partial}^k, (\Delta^\alpha \underline{x})^{\otimes^{k-1}} \right\rangle_{k-1} \; \forall \, \underline{x} \in \Omega'_\alpha$$

so that we have

$$\begin{aligned} \tau^\alpha_{ij}\partial_j &= 0, \\ \tau^\alpha_{ij}\partial^2_{jk_1}\Delta^\alpha x_{k_1} &= 0, \\ \tau^\alpha_{ij}\partial^3_{jk_1k_2}\Delta^\alpha x_{k_1}\Delta^\alpha x_{k_2} &= 0, \\ \vdots \quad &= \vdots \\ \tau^\alpha_{ij}\partial^p_{jk_1\ldots k_{p-1}}\Delta^\alpha x_{k_1}\ldots\Delta^\alpha x_{k_{p-1}} &= 0. \end{aligned}$$

- Due to continuity of polarization field, we have $\partial^k_{k_1k_2\ldots k_k}\tau^\alpha_{ij} = \partial^k_{k_1^*k_2^*\ldots k_k^*}\tau^\alpha_{ij}$ for every permutation $(k_1^*, k_2^*, \ldots, k_k^*)$ of $(k_1, k_2, \ldots, k_k)$
- Then, we enforce equilibrium as follows:

$$\begin{cases} \tau^\alpha_{1j}\partial_j = 0 \\ \tau^\alpha_{2j}\partial_j = 0 \end{cases} \iff \begin{cases} \boxed{\tau^\alpha_{11}\partial_1} + \tau^\alpha_{12}\partial_2 = 0 \\ \tau^\alpha_{12}\partial_1 + \boxed{\tau^\alpha_{22}\partial_2} = 0 \end{cases}$$

$$\begin{cases} \tau^\alpha_{1j}\partial^2_{j1} = 0 \\ \tau^\alpha_{1j}\partial^2_{j2} = 0 \\ \tau^\alpha_{2j}\partial^2_{j1} = 0 \\ \tau^\alpha_{2j}\partial^2_{j2} = 0 \end{cases} \iff \begin{cases} \boxed{\tau^\alpha_{11}\partial^2_{11}} + \tau^\alpha_{12}\partial^2_{21} = 0 \\ \boxed{\tau^\alpha_{11}\partial^2_{12}} + \tau^\alpha_{12}\partial^2_{22} = 0 \\ \tau^\alpha_{12}\partial^2_{11} + \boxed{\tau^\alpha_{22}\partial^2_{21}} = 0 \\ \tau^\alpha_{12}\partial^2_{12} + \boxed{\tau^\alpha_{22}\partial^2_{22}} = 0 \end{cases}$$

$$\begin{cases} \tau^\alpha_{1j}\partial^3_{j11} = 0 \\ \tau^\alpha_{1j}\partial^3_{j22} = 0 \\ \tau^\alpha_{1j}\partial^3_{j12} = 0 \\ \tau^\alpha_{2j}\partial^3_{j11} = 0 \\ \tau^\alpha_{2j}\partial^3_{j22} = 0 \\ \tau^\alpha_{2j}\partial^3_{j12} = 0 \end{cases} \iff \begin{cases} \boxed{\tau^\alpha_{11}\partial^3_{111}} + \tau^\alpha_{12}\partial^3_{211} = 0 \\ \boxed{\tau^\alpha_{11}\partial^3_{122}} + \tau^\alpha_{12}\partial^3_{222} = 0 \\ \boxed{\tau^\alpha_{11}\partial^3_{112}} + \tau^\alpha_{12}\partial^3_{212} = 0 \\ \tau^\alpha_{12}\partial^3_{111} + \boxed{\tau^\alpha_{22}\partial^3_{211}} = 0 \\ \tau^\alpha_{12}\partial^3_{122} + \boxed{\tau^\alpha_{22}\partial^3_{222}} = 0 \\ \tau^\alpha_{12}\partial^3_{112} + \boxed{\tau^\alpha_{22}\partial^3_{212}} = 0 \end{cases}$$

$\ldots$

What about local equilibrium of the polarization field?

- Consequently, we intend to compute

$$\{\tau_{11}^{\alpha}\partial_1, \tau_{11}^{\alpha}\partial_2, \tau_{22}^{\alpha}\partial_1, \tau_{22}^{\alpha}\partial_2\}$$
$$\{\tau_{11}^{\alpha}\partial_{11}^2, \tau_{11}^{\alpha}\partial_{22}^2, \tau_{11}^{\alpha}\partial_{12}^2, \tau_{22}^{\alpha}\partial_{11}^2, \tau_{22}^{\alpha}\partial_{22}^2, \tau_{22}^{\alpha}\partial_{12}^2\}$$
$$\{\tau_{11}^{\alpha}\partial_{111}^3, \tau_{11}^{\alpha}\partial_{222}^3, \tau_{11}^{\alpha}\partial_{112}^3, \tau_{11}^{\alpha}\partial_{221}^3, \tau_{22}^{\alpha}\partial_{111}^3, \tau_{22}^{\alpha}\partial_{222}^3, \tau_{22}^{\alpha}\partial_{112}^3, \tau_{22}^{\alpha}\partial_{221}^3\}$$
$$\vdots$$

  by solving for a stationary state of the HS functional, and...

- Compute

$$\{\tau_{12}^{\alpha}\partial_1, \tau_{12}^{\alpha}\partial_2\}$$
$$\{\tau_{12}^{\alpha}\partial_{11}^2, \tau_{12}^{\alpha}\partial_{22}^2, \tau_{12}^{\alpha}\partial_{12}^2\}$$
$$\{\tau_{12}^{\alpha}\partial_{111}^3, \tau_{12}^{\alpha}\partial_{222}^3, \tau_{12}^{\alpha}\partial_{112}^3, \tau_{12}^{\alpha}\partial_{221}^3\}$$
$$\vdots$$

  from local equilibrium constraints (see previous slides).

HS functional for trial fields in $\mathcal{V}^{h_p}$ (derivation)

From our definition of the estimates of influence tensors, we obtain

$$\overline{\boldsymbol{\tau}^{h_p} : {}^n(\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_p})} = \tau^{\alpha}_{ij}({}^nT^{\alpha\gamma}_{0,0})_{ijkl}\tau^{\gamma}_{kl} + \tau^{\alpha}_{ij}({}^nT^{\alpha\gamma}_{0,1})_{ijkls_1}\partial_{s_1}\tau^{\gamma}_{kl} + \tau^{\alpha}_{ij}({}^nT^{\alpha\gamma}_{0,2})_{ijkls_1s_2}\partial^2_{s_1s_2kl}\tau^{\gamma} + \dots$$

$$+\partial_{r_1}\tau^{\gamma}_{ij}({}^nT^{\alpha\gamma}_{1,0})_{r_1ijkl}\tau^{\gamma}_{kl} + \partial_{r_1}\tau^{\gamma}_{ij}({}^nT^{\alpha\gamma}_{1,1})_{r_1ijkls_1}\partial_{s_1}\tau^{\gamma}_{kl} + \partial_{r_1}\tau^{\gamma}_{ij}({}^nT^{\alpha\gamma}_{1,2})_{r_1ijkls_1s_2}\partial^2_{s_1s_2}\tau^{\gamma}_{kl} + \dots$$

$$+\partial^2_{r_1r_2}\tau^{\gamma}_{ij}({}^nT^{\alpha\gamma}_{1,0})_{r_1r_2ijkl}\tau^{\gamma}_{kl} + \partial^2_{r_1r_2}\tau^{\gamma}_{ij}({}^nT^{\alpha\gamma}_{1,1})_{r_1r_2ijkls_1}\partial_{s_1}\tau^{\gamma}_{kl} + \partial^2_{r_1r_2}\tau^{\gamma}_{ij}({}^nT^{\alpha\gamma}_{1,2})_{r_1r_2ijkls_1s_2}\partial^2_{s_1s_2}\tau^{\gamma}_{kl} + \dots$$

$+\cdots$ which we recast in

$$\overline{\boldsymbol{\tau}^{h_p} : {}^n(\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_p})} = \sum_{\alpha}\sum_{\gamma}\left[\boldsymbol{\tau}^{\alpha} : {}^n\mathbb{T}^{\alpha\gamma}_{0,0} : \boldsymbol{\tau}^{\gamma} + \sum_{r=1}^{p}\sum_{s=1}^{p}\left\langle \boldsymbol{\partial}^r\boldsymbol{\tau}^{\alpha}, \left\langle {}^n\mathbb{T}^{\alpha\gamma}_{r,s}, \boldsymbol{\tau}^{\gamma}\boldsymbol{\partial}^s\right\rangle_{s+2}\right\rangle_{r+2}\right]$$

The other term, $\overline{\boldsymbol{\tau}^{h_p} : (\Delta\mathbb{L})^{-1} : \boldsymbol{\tau}^{h_p}}$ can be calculated exactly:

After change of variables, we have:

$$\overline{\boldsymbol{\tau}^{h_p} : (\Delta\mathbb{L})^{-1} : \boldsymbol{\tau}^{h_p}} = \tau^{\alpha}_{ij}\int_{\Omega'_\alpha}(\Delta L^{\alpha})^{-1}_{ijkl}\mathrm{d}\nu_{\underline{x}}\tau^{\alpha}_{kl} + \tau^{\alpha}_{ij}\int_{\Omega'_\alpha}(\Delta L^{\alpha})^{-1}_{ijkl}x_r\mathrm{d}\nu_{\underline{x}}\partial_r\tau^{\alpha}_{kl} + \tau^{\alpha}_{ij}\int_{\Omega'_\alpha}(\Delta L^{\alpha})^{-1}_{ijkl}x_rx_s\mathrm{d}\nu_{\underline{x}}\partial^2_{rs}\tau^{\alpha}_{kl} + \dots$$

$$+\partial_r\tau^{\alpha}_{ij}\int_{\Omega'_\alpha}x_r(\Delta L^{\alpha})^{-1}_{ijkl}\mathrm{d}\nu_{\underline{x}}\tau^{\alpha}_{kl} + \partial_r\tau^{\alpha}_{ij}\int_{\Omega'_\alpha}x_r(\Delta L^{\alpha})^{-1}_{ijkl}x_s\mathrm{d}\nu_{\underline{x}}\partial_s\tau^{\alpha}_{kl} + \partial_r\tau^{\alpha}_{ij}\int_{\Omega'_\alpha}x_r(\Delta L^{\alpha})^{-1}_{ijkl}x_sx_t\mathrm{d}\nu_{\underline{x}}\partial^2_{st}\tau^{\alpha}_{kl} + \dots$$

$$+\partial^2_{rs}\tau^{\alpha}_{ij}\int_{\Omega'_\alpha}x_rx_s(\Delta L^{\alpha})^{-1}_{ijkl}\mathrm{d}\nu_{\underline{x}}\tau^{\alpha}_{kl} + \partial^2_{rs}\tau^{\alpha}_{ij}\int_{\Omega'_\alpha}x_rx_s(\Delta L^{\alpha})^{-1}_{ijkl}x_t\mathrm{d}\nu_{\underline{x}}\partial_s\tau^{\alpha}_{kl} + \partial^2_{rs}\tau^{\alpha}_{ij}\int_{\Omega'_\alpha}x_rx_s(\Delta L^{\alpha})^{-1}_{ijkl}x_tx_u\mathrm{d}\nu_{\underline{x}}\partial^2_{tu}\tau^{\alpha}_{kl} + \dots$$

$+\dots$

HS functional for trial fields in $\mathcal{V}^{h_p}$

... which we recast in

$$\overline{\boldsymbol{\tau}^{h_p} : (\Delta\mathbb{L})^{-1} : \boldsymbol{\tau}^{h_p}} = \sum_{\alpha} \Delta\mathbb{M}^{\alpha} :: \left[c_\alpha \boldsymbol{\tau}^{\alpha} \otimes \boldsymbol{\tau}^{\alpha} + \sum_{r=1}^{p} \sum_{s=1}^{p} \left\langle \boldsymbol{\tau}^{\alpha} \boldsymbol{\partial}^{r}, \left\langle \mathcal{W}_0^{r+s,0}(\Omega'_\alpha), \boldsymbol{\partial}^{s} \boldsymbol{\tau}^{\alpha} \right\rangle_s \right\rangle_r \right]$$

where $\Delta\mathbb{M}^{\alpha} := (\mathbb{L}^{\alpha} - \mathbb{L}^{0})^{-1}$ so that the following estimate of the HS functional is obtained

$${}^{n}\mathcal{H}(\boldsymbol{\tau}^{h_p}) := \overline{\boldsymbol{\tau}^{h_p}} : \overline{\boldsymbol{\varepsilon}} - 1/2\overline{\boldsymbol{\tau}^{h_p} : (\Delta\mathbb{L})^{-1} : \boldsymbol{\tau}^{h_p}} - 1/2\overline{\boldsymbol{\tau}^{h_p} : {}^{n}(\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_p})}$$

so that we have

$$\begin{aligned} {}^{n}\mathcal{H}(\boldsymbol{\tau}^{h_p}) = \sum_{\alpha} & \left(c_\alpha \boldsymbol{\tau}^{\alpha} : \overline{\boldsymbol{\varepsilon}} + \sum_{r=1}^{p} \left\langle \boldsymbol{\tau}^{\alpha} \boldsymbol{\partial}^{r}, \mathcal{W}_0^{r,0}(\Omega'_\alpha) \right\rangle_r : \overline{\boldsymbol{\varepsilon}} \right) \\ & - \frac{1}{2} \sum_{\alpha} \Delta\mathbb{M}^{\alpha} :: \left(c_\alpha \boldsymbol{\tau}^{\alpha} \otimes \boldsymbol{\tau}^{\alpha} + \sum_{r=1}^{p} \sum_{s=1}^{p} \left\langle \boldsymbol{\tau}^{\alpha} \boldsymbol{\partial}^{r}, \left\langle \mathcal{W}_0^{r+s,0}(\Omega'_\alpha), \boldsymbol{\partial}^{s} \boldsymbol{\tau}^{\alpha} \right\rangle_s \right\rangle_r \right) \\ & - \frac{1}{2} \sum_{\alpha} \sum_{\gamma} \left(\boldsymbol{\tau}^{\alpha} : {}^{n}\mathbb{T}_{0,0}^{\alpha\gamma} : \boldsymbol{\tau}^{\gamma} + \sum_{r=1}^{p} \sum_{s=1}^{p} \left\langle \boldsymbol{\partial}^{r} \boldsymbol{\tau}^{\alpha}, \left\langle {}^{n}\mathbb{T}_{r,s}^{\alpha\gamma}, \boldsymbol{\tau}^{\gamma} \boldsymbol{\partial}^{s} \right\rangle_{s+2} \right\rangle_{r+2} \right) \end{aligned}$$

Now, we want to solve for the stationary state of the functional, i.e. find $\{\boldsymbol{\tau}^{\alpha}, \boldsymbol{\partial}^{r}\boldsymbol{\tau}^{\alpha} \mid 1 \leq r \leq p\}$ for all α s.t. ${}^{n}\mathcal{H}(\boldsymbol{\tau}^{h_p})$ is optimized.

2D Formalism

- Generalized Mandel notation
- Solution of global stationarity equations
- 2D Stroh formalism
- 2D integral Barnett-Lothe formalism

"Generalized Mandel representation" for assembly of a global system of stationarity equations

The components of $\boldsymbol{\partial}^r \boldsymbol{\tau}$ are stored into vectors of the form

$r = 0$

$$\{\boldsymbol{\tau}\} := \begin{bmatrix} \tau^1_{11} \\ \tau^1_{22} \\ \sqrt{2}\tau^1_{12} \\ \tau^2_{11} \\ \tau^2_{22} \\ \sqrt{2}\tau^2_{12} \\ \vdots \end{bmatrix}$$

$$3(r+1)n_\alpha \times 1 = 3n_\alpha \times 1$$

$r = 1$

$$\{\boldsymbol{\partial\tau}\} := \begin{bmatrix} \partial_1\tau^1_{11} \\ \partial_1\tau^1_{22} \\ \sqrt{2}\partial_1\tau^1_{12} \\ \partial_2\tau^1_{11} \\ \partial_2\tau^1_{22} \\ \sqrt{2}\partial_2\tau^1_{12} \\ \partial_1\tau^2_{11} \\ \partial_1\tau^2_{22} \\ \sqrt{2}\partial_1\tau^2_{12} \\ \partial_2\tau^2_{11} \\ \partial_2\tau^2_{22} \\ \sqrt{2}\partial_2\tau^2_{12} \\ \vdots \end{bmatrix}$$

$$3(r+1)n_\alpha \times 1 = 6n_\alpha \times 1$$

$r = 2$

$$\{\boldsymbol{\partial}^2\boldsymbol{\tau}\} := \begin{bmatrix} \partial^2_{11}\tau^1_{11} \\ \partial^2_{11}\tau^1_{22} \\ \sqrt{2}\partial^2_{11}\tau^1_{12} \\ \partial^2_{22}\tau^1_{11} \\ \partial^2_{22}\tau^1_{22} \\ \sqrt{2}\partial^2_{22}\tau^1_{12} \\ \sqrt{2}\partial^2_{12}\tau^1_{11} \\ \sqrt{2}\partial^2_{12}\tau^1_{22} \\ \sqrt{2}\partial^2_{12}\tau^1_{12} \\ \partial^2_{11}\tau^2_{11} \\ \partial^2_{11}\tau^2_{22} \\ \sqrt{2}\partial^2_{11}\tau^2_{12} \\ \partial^2_{22}\tau^2_{11} \\ \partial^2_{22}\tau^2_{22} \\ \sqrt{2}\partial^2_{22}\tau^2_{12} \\ \sqrt{2}\partial^2_{12}\tau^2_{11} \\ \sqrt{2}\partial^2_{12}\tau^2_{22} \\ 2\partial^2_{12}\tau^2_{12} \\ \vdots \end{bmatrix}$$

$$3(r+1)n_\alpha \times 1 = 9n_\alpha \times 1$$

$r = 3$

$$\{\boldsymbol{\partial}^3\boldsymbol{\tau}\} := \begin{bmatrix} \partial^3_{111}\tau^1_{11} \\ \partial^3_{111}\tau^1_{22} \\ \sqrt{2}\partial^3_{111}\tau^1_{12} \\ \partial^3_{222}\tau^1_{11} \\ \partial^3_{222}\tau^1_{22} \\ \sqrt{2}\partial^3_{222}\tau^1_{12} \\ \sqrt{3}\partial^3_{112}\tau^1_{11} \\ \sqrt{3}\partial^3_{112}\tau^1_{22} \\ \sqrt{6}\partial^3_{112}\tau^1_{12} \\ \sqrt{3}\partial^3_{122}\tau^1_{11} \\ \sqrt{3}\partial^3_{122}\tau^1_{22} \\ \sqrt{6}\partial^3_{122}\tau^1_{12} \\ \partial^3_{111}\tau^2_{11} \\ \partial^3_{111}\tau^2_{22} \\ \sqrt{2}\partial^3_{111}\tau^2_{12} \\ \partial^3_{222}\tau^2_{11} \\ \partial^3_{222}\tau^2_{22} \\ \sqrt{2}\partial^3_{222}\tau^2_{12} \\ \sqrt{3}\partial^3_{112}\tau^2_{11} \\ \sqrt{3}\partial^3_{112}\tau^2_{22} \\ \sqrt{6}\partial^3_{112}\tau^2_{12} \\ \sqrt{3}\partial^3_{122}\tau^2_{11} \\ \sqrt{3}\partial^3_{122}\tau^2_{22} \\ \sqrt{6}\partial^3_{122}\tau^2_{12} \\ \vdots \end{bmatrix}$$

$$3(r+1)n_\alpha \times 1 = 12n_\alpha \times 1$$

$r = 4$

$$\{\boldsymbol{\partial}^4\boldsymbol{\tau}\} := \begin{bmatrix} \partial^4_{1111}\tau^1_{11} \\ \partial^4_{1111}\tau^1_{22} \\ \sqrt{2}\partial^4_{1111}\tau^1_{12} \\ \partial^4_{2222}\tau^1_{11} \\ \partial^4_{2222}\tau^1_{22} \\ \sqrt{2}\partial^4_{2222}\tau^1_{12} \\ \sqrt{4}\partial^4_{1112}\tau^1_{11} \\ \sqrt{4}\partial^4_{1112}\tau^1_{22} \\ \sqrt{8}\partial^4_{1112}\tau^1_{12} \\ \sqrt{6}\partial^4_{1122}\tau^1_{11} \\ \sqrt{6}\partial^4_{1122}\tau^1_{22} \\ \sqrt{12}\partial^4_{1122}\tau^1_{12} \\ \sqrt{4}\partial^4_{1222}\tau^1_{11} \\ \sqrt{4}\partial^4_{1222}\tau^1_{22} \\ \sqrt{8}\partial^4_{1222}\tau^1_{12} \\ \partial^4_{1111}\tau^2_{11} \\ \partial^4_{1111}\tau^2_{22} \\ \sqrt{2}\partial^4_{1111}\tau^2_{12} \\ \partial^4_{2222}\tau^2_{11} \\ \partial^4_{2222}\tau^2_{22} \\ \sqrt{2}\partial^4_{2222}\tau^2_{12} \\ \sqrt{4}\partial^4_{1112}\tau^2_{11} \\ \sqrt{4}\partial^4_{1112}\tau^2_{22} \\ \sqrt{8}\partial^4_{1112}\tau^2_{12} \\ \sqrt{6}\partial^4_{1122}\tau^2_{11} \\ \sqrt{6}\partial^4_{1122}\tau^2_{22} \\ \sqrt{12}\partial^4_{1122}\tau^2_{12} \\ \sqrt{4}\partial^4_{1222}\tau^2_{11} \\ \sqrt{4}\partial^4_{1222}\tau^2_{22} \\ \sqrt{8}\partial^4_{1222}\tau^2_{12} \\ \vdots \end{bmatrix} \quad \cdots$$

$$3(r+1)n_\alpha \times 1 = 15n_\alpha \times 1$$

“Generalized Mandel representation” for assembly of a global system of stationarity equations

The components of $\overline{\boldsymbol{\varepsilon}} \otimes \mathcal{W}_0^{r,0}(\Omega'_\alpha)$ are stored into vectors of the form

$r=0$:

$$\{\overline{\varepsilon}^0\} := \begin{bmatrix} c_1\overline{\varepsilon}_{11} \\ c_1\overline{\varepsilon}_{22} \\ \sqrt{2}c_1\overline{\varepsilon}_{12} \\ c_2\overline{\varepsilon}_{11} \\ c_2\overline{\varepsilon}_{22} \\ \sqrt{2}c_2\overline{\varepsilon}_{12} \\ \vdots \end{bmatrix}$$

$3n_\alpha \times 1$

$r=1$:

$$\{\overline{\varepsilon}^1\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_1))_1 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_1))_1 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_1))_1 \\ \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_1))_2 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_1))_2 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_1))_2 \\ \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_2))_1 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_2))_1 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_2))_1 \\ \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_2))_2 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_2))_2 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_2))_2 \\ \vdots \end{bmatrix}$$

$6n_\alpha \times 1$

$r=2$:

$$\{\overline{\varepsilon}^2\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_1))_{11} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_1))_{11} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_1))_{11} \\ \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_1))_{22} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_1))_{22} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_1))_{22} \\ \sqrt{2}\overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_1))_{12} \\ \sqrt{2}\overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_1))_{12} \\ \sqrt{4}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_1))_{12} \\ \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_2))_{11} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_2))_{11} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_2))_{11} \\ \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_2))_{22} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_2))_{22} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_2))_{22} \\ \sqrt{2}\overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_2))_{12} \\ \sqrt{2}\overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_2))_{12} \\ \sqrt{4}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_2))_{12} \\ \vdots \end{bmatrix}$$

$9n_\alpha \times 1$

$r=3$:

$$\{\overline{\varepsilon}^3\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{111} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{111} \\ \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{222} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{222} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{112} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{112} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{112} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{122} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{122} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{122} \\ \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{111} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{111} \\ \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{222} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{222} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{112} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{112} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{112} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{122} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{122} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{122} \\ \vdots \end{bmatrix}$$

$12n_\alpha \times 1$

$r=4$:

$$\{\overline{\varepsilon}^4\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1111} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1111} \\ \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{2222} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{2222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{2222} \\ \sqrt{4}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1112} \\ \sqrt{4}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1112} \\ \sqrt{8}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1112} \\ \sqrt{6}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1122} \\ \sqrt{6}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1122} \\ \sqrt{12}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1122} \\ \sqrt{4}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1222} \\ \sqrt{4}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1222} \\ \sqrt{8}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1222} \\ \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{1111} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{1111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{1111} \\ \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{2222} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{2222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{2222} \\ \sqrt{4}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{1112} \\ \sqrt{4}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{1112} \\ \sqrt{8}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{1112} \\ \sqrt{6}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{1122} \\ \sqrt{6}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{1122} \\ \sqrt{12}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{1122} \\ \vdots \end{bmatrix} \quad \cdots$$

$15n_\alpha \times 1$

"Generalized Mandel representation" for assembly of a global system of stationarity equations

The components of $\partial^r \boldsymbol{\tau}$ are stored into vectors of the form

$r = 0$

$$\{\boldsymbol{\tau}\} := \begin{bmatrix} \tau_{11}^1 \\ \tau_{22}^1 \\ \tau_{11}^2 \\ \tau_{22}^2 \\ \vdots \end{bmatrix}$$

$$2(r+1)n_\alpha \times 1 = 2n_\alpha \times 1$$

$r = 1$

$$\{\boldsymbol{\partial\tau}\} := \begin{bmatrix} \partial_1 \tau_{11}^1 \\ \partial_1 \tau_{22}^1 \\ \partial_2 \tau_{11}^1 \\ \partial_2 \tau_{22}^1 \\ \partial_1 \tau_{11}^2 \\ \partial_1 \tau_{22}^2 \\ \partial_2 \tau_{11}^2 \\ \partial_2 \tau_{22}^2 \\ \vdots \end{bmatrix}$$

$$2(r+1)n_\alpha \times 1 = 4n_\alpha \times 1$$

$r = 2$

$$\{\boldsymbol{\partial}^2\boldsymbol{\tau}\} := \begin{bmatrix} \partial_{11}^2 \tau_{11}^1 \\ \partial_{11}^2 \tau_{22}^1 \\ \partial_{22}^2 \tau_{11}^1 \\ \partial_{22}^2 \tau_{22}^1 \\ \sqrt{2}\partial_{12}^2 \tau_{11}^1 \\ \sqrt{2}\partial_{12}^2 \tau_{22}^1 \\ \partial_{11}^2 \tau_{11}^2 \\ \partial_{11}^2 \tau_{22}^2 \\ \partial_{22}^2 \tau_{11}^2 \\ \partial_{22}^2 \tau_{22}^2 \\ \sqrt{2}\partial_{12}^2 \tau_{11}^2 \\ \sqrt{2}\partial_{12}^2 \tau_{22}^2 \\ 2\partial_{12}^2 \tau_{12}^2 \\ \vdots \end{bmatrix}$$

$$2(r+1)n_\alpha \times 1 = 6n_\alpha \times 1$$

$r = 3$

$$\{\boldsymbol{\partial}^3\boldsymbol{\tau}\} := \begin{bmatrix} \partial_{111}^3 \tau_{11}^1 \\ \partial_{111}^3 \tau_{22}^1 \\ \partial_{222}^3 \tau_{11}^1 \\ \partial_{222}^3 \tau_{22}^1 \\ \sqrt{3}\partial_{112}^3 \tau_{11}^1 \\ \sqrt{3}\partial_{112}^3 \tau_{22}^1 \\ \sqrt{3}\partial_{122}^3 \tau_{11}^1 \\ \sqrt{3}\partial_{122}^3 \tau_{22}^1 \\ \partial_{111}^3 \tau_{11}^2 \\ \partial_{111}^3 \tau_{22}^2 \\ \partial_{222}^3 \tau_{11}^2 \\ \partial_{222}^3 \tau_{22}^2 \\ \sqrt{3}\partial_{112}^3 \tau_{11}^2 \\ \sqrt{3}\partial_{112}^3 \tau_{22}^2 \\ \sqrt{3}\partial_{122}^3 \tau_{11}^2 \\ \sqrt{3}\partial_{122}^3 \tau_{22}^2 \\ \vdots \end{bmatrix}$$

$$2(r+1)n_\alpha \times 1 = 8n_\alpha \times 1$$

$r = 4$

$$\{\boldsymbol{\partial}^4\boldsymbol{\tau}\} := \begin{bmatrix} \partial_{1111}^4 \tau_{11}^1 \\ \partial_{1111}^4 \tau_{22}^1 \\ \partial_{2222}^4 \tau_{11}^1 \\ \partial_{2222}^4 \tau_{22}^1 \\ \sqrt{4}\partial_{1112}^4 \tau_{11}^1 \\ \sqrt{4}\partial_{1112}^4 \tau_{22}^1 \\ \sqrt{6}\partial_{1122}^4 \tau_{11}^1 \\ \sqrt{6}\partial_{1122}^4 \tau_{22}^1 \\ \sqrt{4}\partial_{1222}^4 \tau_{11}^1 \\ \sqrt{4}\partial_{1222}^4 \tau_{22}^1 \\ \partial_{1111}^4 \tau_{11}^2 \\ \partial_{1111}^4 \tau_{22}^2 \\ \partial_{2222}^4 \tau_{11}^2 \\ \partial_{2222}^4 \tau_{22}^2 \\ \sqrt{4}\partial_{1112}^4 \tau_{11}^2 \\ \sqrt{4}\partial_{1112}^4 \tau_{22}^2 \\ \sqrt{6}\partial_{1122}^4 \tau_{11}^2 \\ \sqrt{6}\partial_{1122}^4 \tau_{22}^2 \\ \sqrt{4}\partial_{1222}^4 \tau_{11}^2 \\ \sqrt{4}\partial_{1222}^4 \tau_{22}^2 \\ \vdots \end{bmatrix} \quad \cdots$$

$$2(r+1)n_\alpha \times 1 = 10n_\alpha \times 1$$

Gradient components of the shear trial field are enforced through constraints derived from local equilibrium.

"Generalized Mandel representation" for assembly of a global system of stationarity equations (tri)

The components of $\overline{\boldsymbol{\varepsilon}} \otimes \mathcal{W}_0^{r,0}(\Omega'_\alpha)$ are stored into vectors of the form

$$r = 0 \qquad \{\overline{\boldsymbol{\varepsilon}}^0\} := \begin{bmatrix} c_1\overline{\varepsilon}_{11} \\ c_1\overline{\varepsilon}_{22} \\ \sqrt{2}c_1\overline{\varepsilon}_{12} \\ c_2\overline{\varepsilon}_{11} \\ c_2\overline{\varepsilon}_{22} \\ \sqrt{2}c_2\overline{\varepsilon}_{12} \\ \vdots \end{bmatrix} \quad 3n_\alpha \times 1$$

$$r = 1 \qquad \{\overline{\boldsymbol{\varepsilon}}^1\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_1))_1 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_1))_1 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_1))_1 \\ \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_1))_2 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_1))_2 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_1))_2 \\ \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_2))_1 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_2))_1 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_2))_1 \\ \overline{\varepsilon}_{11}(W_0^{1,0}(\Omega'_2))_2 \\ \overline{\varepsilon}_{22}(W_0^{1,0}(\Omega'_2))_2 \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{1,0}(\Omega'_2))_2 \\ \vdots \end{bmatrix} \quad 6n_\alpha \times 1$$

$$r = 2 \qquad \{\overline{\boldsymbol{\varepsilon}}^2\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_1))_{11} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_1))_{11} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_1))_{11} \\ \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_1))_{22} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_1))_{22} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_1))_{22} \\ \sqrt{2}\overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_1))_{12} \\ \sqrt{2}\overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_1))_{12} \\ \sqrt{4}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_1))_{12} \\ \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_2))_{11} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_2))_{11} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_2))_{11} \\ \overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_2))_{22} \\ \overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_2))_{22} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_2))_{22} \\ \sqrt{2}\overline{\varepsilon}_{11}(W_0^{2,0}(\Omega'_2))_{12} \\ \sqrt{2}\overline{\varepsilon}_{22}(W_0^{2,0}(\Omega'_2))_{12} \\ \sqrt{4}\overline{\varepsilon}_{12}(W_0^{2,0}(\Omega'_2))_{12} \\ \vdots \end{bmatrix} \quad 9n_\alpha \times 1$$

$$r = 3 \qquad \{\overline{\boldsymbol{\varepsilon}}^3\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{111} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{111} \\ \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{222} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{222} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{112} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{112} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{112} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_1))_{122} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_1))_{122} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_1))_{122} \\ \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{111} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{111} \\ \overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{222} \\ \overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{222} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{112} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{112} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{112} \\ \sqrt{3}\overline{\varepsilon}_{11}(W_0^{3,0}(\Omega'_2))_{122} \\ \sqrt{3}\overline{\varepsilon}_{22}(W_0^{3,0}(\Omega'_2))_{122} \\ \sqrt{6}\overline{\varepsilon}_{12}(W_0^{3,0}(\Omega'_2))_{122} \\ \vdots \end{bmatrix} \quad 12n_\alpha \times 1$$

$$r = 4 \qquad \{\overline{\boldsymbol{\varepsilon}}^4\} := \begin{bmatrix} \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1111} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1111} \\ \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{2222} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{2222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{2222} \\ \sqrt{4}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1112} \\ \sqrt{4}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1112} \\ \sqrt{8}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1112} \\ \sqrt{6}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1122} \\ \sqrt{6}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1122} \\ \sqrt{12}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1122} \\ \sqrt{4}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_1))_{1222} \\ \sqrt{4}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_1))_{1222} \\ \sqrt{8}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_1))_{1222} \\ \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{1111} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{1111} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{1111} \\ \overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{2222} \\ \overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{2222} \\ \sqrt{2}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{2222} \\ \sqrt{4}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{1112} \\ \sqrt{4}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{1112} \\ \sqrt{8}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{1112} \\ \sqrt{6}\overline{\varepsilon}_{11}(W_0^{4,0}(\Omega'_2))_{1122} \\ \sqrt{6}\overline{\varepsilon}_{22}(W_0^{4,0}(\Omega'_2))_{1122} \\ \sqrt{12}\overline{\varepsilon}_{12}(W_0^{4,0}(\Omega'_2))_{1122} \\ \vdots \end{bmatrix} \quad 15n_\alpha \times 1 \qquad \cdots$$

"Generalized Mandel representation" for assembly of a global system of stationarity equations

Compliance matrices are defined as follows

$$\underset{3\times 3}{[\Delta\mathbb{M}^\alpha]} := \begin{bmatrix} \Delta M^\alpha_{1111} & \Delta M^\alpha_{1122} & \sqrt{2}\Delta M^\alpha_{1112} \\ \Delta M^\alpha_{2211} & \Delta M^\alpha_{2222} & \sqrt{2}\Delta M^\alpha_{2212} \\ \sqrt{2}\Delta M^\alpha_{1211} & \sqrt{2}\Delta M^\alpha_{1222} & 2\Delta M^\alpha_{1212} \end{bmatrix}$$

so that the components of $\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{s+r,0}$ are stored into matrices $\underset{3(r+1)\times 3(s+1)}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{s+r,0}]}$ defined by

$$\underset{6\times 6}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{1+1,0}]} := \begin{bmatrix} (W_0^{2,0}(\Omega'_\alpha))_{11}[\Delta\mathbb{M}^\alpha] & (W_0^{2,0}(\Omega'_\alpha))_{12}[\Delta\mathbb{M}^\alpha] \\ (W_0^{2,0}(\Omega'_\alpha))_{12}[\Delta\mathbb{M}^\alpha] & (W_0^{2,0}(\Omega'_\alpha))_{22}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{6\times 9}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{2+1,0}]} := \begin{bmatrix} (W_0^{3,0}(\Omega'_\alpha))_{111}[\Delta\mathbb{M}^\alpha] & (W_0^{3,0}(\Omega'_\alpha))_{122}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{3,0}(\Omega'_\alpha))_{112}[\Delta\mathbb{M}^\alpha] \\ (W_0^{3,0}(\Omega'_\alpha))_{112}[\Delta\mathbb{M}^\alpha] & (W_0^{3,0}(\Omega'_\alpha))_{222}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{3,0}(\Omega'_\alpha))_{122}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{6\times 12}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{3+1,0}]} := \begin{bmatrix} (W_0^{4,0}(\Omega'_\alpha))_{1111}[\Delta\mathbb{M}^\alpha] & (W_0^{4,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{4,0}(\Omega'_\alpha))_{1112}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{4,0}(\Omega'_\alpha))_{1122}[\Delta\mathbb{M}^\alpha] \\ (W_0^{4,0}(\Omega'_\alpha))_{1112}[\Delta\mathbb{M}^\alpha] & (W_0^{4,0}(\Omega'_\alpha))_{2222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{4,0}(\Omega'_\alpha))_{1122}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{4,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{6\times 15}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{4+1,0}]} := \begin{bmatrix} (W_0^{5,0}(\Omega'_\alpha))_{11111}[\Delta\mathbb{M}^\alpha] & (W_0^{5,0}(\Omega'_\alpha))_{12222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{5,0}(\Omega'_\alpha))_{11112}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] \\ (W_0^{5,0}(\Omega'_\alpha))_{11112}[\Delta\mathbb{M}^\alpha] & (W_0^{5,0}(\Omega'_\alpha))_{22222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{5,0}(\Omega'_\alpha))_{12222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

`dMW_local(α,s=1,r=2,I=6…8,J=0…3)`

$$\underset{9\times 6}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{1+2,0}]} := \begin{bmatrix} (W_0^{3,0}(\Omega'_\alpha))_{111}[\Delta\mathbb{M}^\alpha] & (W_0^{3,0}(\Omega'_\alpha))_{112}[\Delta\mathbb{M}^\alpha] \\ (W_0^{3,0}(\Omega'_\alpha))_{122}[\Delta\mathbb{M}^\alpha] & (W_0^{3,0}(\Omega'_\alpha))_{222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{2}(W_0^{3,0}(\Omega'_\alpha))_{112}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{3,0}(\Omega'_\alpha))_{122}[\Delta\mathbb{M}^\alpha] \end{bmatrix} = [\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{2+1,0}]^T$$

$$\underset{9\times 9}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{2+2,0}]} := \begin{bmatrix} (W_0^{4,0}(\Omega'_\alpha))_{1111}[\Delta\mathbb{M}^\alpha] & (W_0^{4,0}(\Omega'_\alpha))_{1122}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1112}[\Delta\mathbb{M}^\alpha] \\ (W_0^{4,0}(\Omega'_\alpha))_{1122}[\Delta\mathbb{M}^\alpha] & (W_0^{4,0}(\Omega'_\alpha))_{2222}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1112}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{4,0}(\Omega'_\alpha))_{1122}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{9\times 12}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{3+2,0}]} := \begin{bmatrix} (W_0^{5,0}(\Omega'_\alpha))_{11111}[\Delta\mathbb{M}^\alpha] & (W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{11112}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] \\ (W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] & (W_0^{5,0}(\Omega'_\alpha))_{22222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{12222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{2}(W_0^{5,0}(\Omega'_\alpha))_{11112}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{5,0}(\Omega'_\alpha))_{12222}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{9\times 15}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{4+2,0}]} := \begin{bmatrix} (W_0^{6,0}(\Omega'_\alpha))_{111111}[\Delta\mathbb{M}^\alpha] & (W_0^{6,0}(\Omega'_\alpha))_{112222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{6,0}(\Omega'_\alpha))_{111112}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{6,0}(\Omega'_\alpha))_{111122}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{6,0}(\Omega'_\alpha))_{111222}[\Delta\mathbb{M}^\alpha] \\ (W_0^{6,0}(\Omega'_\alpha))_{111122}[\Delta\mathbb{M}^\alpha] & (W_0^{6,0}(\Omega'_\alpha))_{222222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{6,0}(\Omega'_\alpha))_{111222}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{6,0}(\Omega'_\alpha))_{112222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{6,0}(\Omega'_\alpha))_{122222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{2}(W_0^{6,0}(\Omega'_\alpha))_{111112}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{6,0}(\Omega'_\alpha))_{122222}[\Delta\mathbb{M}^\alpha] & \sqrt{8}(W_0^{6,0}(\Omega'_\alpha))_{111122}[\Delta\mathbb{M}^\alpha] & \sqrt{12}(W_0^{6,0}(\Omega'_\alpha))_{111222}[\Delta\mathbb{M}^\alpha] & \sqrt{8}(W_0^{6,0}(\Omega'_\alpha))_{112222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

"Generalized Mandel representation" for assembly of a global system of stationarity equations

so that the components of $\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{s+r,0}$ are stored into matrices $[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{s+r,0}]$ defined by

$3(r+1)\times 3(s+1)$

$$[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{r+s,0}] = [\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{s+r,0}]^T \ \forall r,s$$

$$r=s \implies [\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{s+r,0}] = [\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{s+r,0}]^T$$

$\Downarrow$

$$[\mathbb{M}_{s,r}] = [\mathbb{M}_{r,s}]^T \ \forall r,s$$

$$[\mathbb{M}_{s,r}] = [\mathbb{M}_{s,r}]^T \impliedby r=s$$

$$\underset{12\times 6}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{1+3,0}]} := \begin{bmatrix} (W_0^{4,0}(\Omega'_\alpha))_{1111}[\Delta\mathbb{M}^\alpha] & (W_0^{4,0}(\Omega'_\alpha))_{1112}[\Delta\mathbb{M}^\alpha] \\ (W_0^{4,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] & (W_0^{4,0}(\Omega'_\alpha))_{2222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1112}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1122}[\Delta\mathbb{M}^\alpha] \\ \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1122}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{4,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

`dMW_local(` α `,s=2,r=3,I=3…5,J=0…3)`

$$\underset{12\times 9}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{2+3,0}]} := \begin{bmatrix} (W_0^{5,0}(\Omega'_\alpha))_{11111}[\Delta\mathbb{M}^\alpha] & (W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{5,0}(\Omega'_\alpha))_{11112}[\Delta\mathbb{M}^\alpha] \\ (W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] & (W_0^{5,0}(\Omega'_\alpha))_{22222}[\Delta\mathbb{M}^\alpha] & \sqrt{2}(W_0^{5,0}(\Omega'_\alpha))_{12222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{11112}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] \\ \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{11122}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{5,0}(\Omega'_\alpha))_{12222}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{5,0}(\Omega'_\alpha))_{11222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{12\times 12}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{3+3,0}]} := \begin{bmatrix} (W_0^{6,0}(\Omega'_\alpha))_{111111}[\Delta\mathbb{M}^\alpha] & (W_0^{6,0}(\Omega'_\alpha))_{111222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{111112}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{111122}[\Delta\mathbb{M}^\alpha] \\ (W_0^{6,0}(\Omega'_\alpha))_{111222}[\Delta\mathbb{M}^\alpha] & (W_0^{6,0}(\Omega'_\alpha))_{222222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{112222}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{122222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{111112}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{112222}[\Delta\mathbb{M}^\alpha] & \sqrt{9}(W_0^{6,0}(\Omega'_\alpha))_{111122}[\Delta\mathbb{M}^\alpha] & \sqrt{9}(W_0^{6,0}(\Omega'_\alpha))_{111222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{111122}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{6,0}(\Omega'_\alpha))_{122222}[\Delta\mathbb{M}^\alpha] & \sqrt{9}(W_0^{6,0}(\Omega'_\alpha))_{111222}[\Delta\mathbb{M}^\alpha] & \sqrt{9}(W_0^{6,0}(\Omega'_\alpha))_{112222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{12\times 15}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{4+3,0}]} := \begin{bmatrix} (W_0^{7,0}(\Omega'_\alpha))_{1111111}[\Delta\mathbb{M}^\alpha] & (W_0^{7,0}(\Omega'_\alpha))_{1112222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{7,0}(\Omega'_\alpha))_{1111112}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{7,0}(\Omega'_\alpha))_{1111122}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{7,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] \\ (W_0^{7,0}(\Omega'_\alpha))_{1111222}[\Delta\mathbb{M}^\alpha] & (W_0^{7,0}(\Omega'_\alpha))_{2222222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{7,0}(\Omega'_\alpha))_{1112222}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{7,0}(\Omega'_\alpha))_{1122222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{7,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{3}(W_0^{7,0}(\Omega'_\alpha))_{1111112}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{7,0}(\Omega'_\alpha))_{1122222}[\Delta\mathbb{M}^\alpha] & \sqrt{12}(W_0^{7,0}(\Omega'_\alpha))_{1111122}[\Delta\mathbb{M}^\alpha] & \sqrt{18}(W_0^{7,0}(\Omega'_\alpha))_{1111222}[\Delta\mathbb{M}^\alpha] & \sqrt{12}(W_0^{7,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{3}(W_0^{7,0}(\Omega'_\alpha))_{1111122}[\Delta\mathbb{M}^\alpha] & \sqrt{3}(W_0^{7,0}(\Omega'_\alpha))_{1222222}[\Delta\mathbb{M}^\alpha] & \sqrt{12}(W_0^{7,0}(\Omega'_\alpha))_{1111222}[\Delta\mathbb{M}^\alpha] & \sqrt{18}(W_0^{7,0}(\Omega'_\alpha))_{1112222}[\Delta\mathbb{M}^\alpha] & \sqrt{12}(W_0^{7,0}(\Omega'_\alpha))_{1222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

$$\underset{15\times 15}{[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}_0^{4+4,0}]} := \begin{bmatrix} (W_0^{8,0}(\Omega'_\alpha))_{11111111}[\Delta\mathbb{M}^\alpha] & (W_0^{8,0}(\Omega'_\alpha))_{11112222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{11111112}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{8,0}(\Omega'_\alpha))_{11111122}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{11111222}[\Delta\mathbb{M}^\alpha] \\ (W_0^{8,0}(\Omega'_\alpha))_{11112222}[\Delta\mathbb{M}^\alpha] & (W_0^{8,0}(\Omega'_\alpha))_{22222222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{11122222}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{8,0}(\Omega'_\alpha))_{11222222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{12222222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{11111112}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{11122222}[\Delta\mathbb{M}^\alpha] & \sqrt{16}(W_0^{8,0}(\Omega'_\alpha))_{11111122}[\Delta\mathbb{M}^\alpha] & \sqrt{24}(W_0^{8,0}(\Omega'_\alpha))_{11111222}[\Delta\mathbb{M}^\alpha] & \sqrt{16}(W_0^{8,0}(\Omega'_\alpha))_{11112222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{6}(W_0^{8,0}(\Omega'_\alpha))_{11111122}[\Delta\mathbb{M}^\alpha] & \sqrt{6}(W_0^{8,0}(\Omega'_\alpha))_{11222222}[\Delta\mathbb{M}^\alpha] & \sqrt{24}(W_0^{8,0}(\Omega'_\alpha))_{11111222}[\Delta\mathbb{M}^\alpha] & \sqrt{36}(W_0^{8,0}(\Omega'_\alpha))_{11112222}[\Delta\mathbb{M}^\alpha] & \sqrt{24}(W_0^{8,0}(\Omega'_\alpha))_{11122222}[\Delta\mathbb{M}^\alpha] \\ \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{11111222}[\Delta\mathbb{M}^\alpha] & \sqrt{4}(W_0^{8,0}(\Omega'_\alpha))_{12222222}[\Delta\mathbb{M}^\alpha] & \sqrt{16}(W_0^{8,0}(\Omega'_\alpha))_{11112222}[\Delta\mathbb{M}^\alpha] & \sqrt{24}(W_0^{8,0}(\Omega'_\alpha))_{11122222}[\Delta\mathbb{M}^\alpha] & \sqrt{16}(W_0^{8,0}(\Omega'_\alpha))_{11222222}[\Delta\mathbb{M}^\alpha] \end{bmatrix}$$

and global Minkowski-weighted compliance matrices are constructed as follows

$$[\mathbb{M}_{0,0}] := \begin{bmatrix} c_0[\Delta\mathbb{M}^0] & 0 & \cdots & 0 \\ 0 & c_1[\Delta\mathbb{M}^1] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n_\alpha-1}[\Delta\mathbb{M}^{n_\alpha-1}] \end{bmatrix}$$

$3n_\alpha \times 3n_\alpha$

and

$$[\mathbb{M}_{s,r}] := \begin{bmatrix} [\Delta\mathbb{M}^0 \otimes \mathcal{W}_0^{s+r,0}] & 0 & \cdots & 0 \\ 0 & [\Delta\mathbb{M}^1 \otimes \mathcal{W}_0^{s+r,0}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [\Delta\mathbb{M}^{n_\alpha-1} \otimes \mathcal{W}_0^{s+r,0}] \end{bmatrix}$$

$3n_\alpha(r+1)\times 3n_\alpha(s+1)$

(for *r>0* and *s>0*).

“Generalized Mandel representation” for assembly of a global system of stationarity equations

- The components of the influence tensors ${}^n\mathbb{T}^{\gamma\alpha}_{0,0}$ are stored into matrices of the form / $r=0$

$$\underset{3\times 3}{\left[{}^n\mathbb{T}^{\gamma\alpha}_{0,0}\right]} := \begin{bmatrix} ({}^nT^{\gamma\alpha}_{0,0})_{1111} & ({}^nT^{\gamma\alpha}_{0,0})_{1122} & \sqrt{2}({}^nT^{\gamma\alpha}_{0,0})_{1112} \\ ({}^nT^{\gamma\alpha}_{0,0})_{2211} & ({}^nT^{\gamma\alpha}_{0,0})_{2222} & \sqrt{2}({}^nT^{\gamma\alpha}_{0,0})_{2212} \\ \sqrt{2}({}^nT^{\gamma\alpha}_{0,0})_{1211} & \sqrt{2}({}^nT^{\gamma\alpha}_{0,0})_{1222} & \sqrt{4}({}^nT^{\gamma\alpha}_{0,0})_{1212} \end{bmatrix}$$

The components of the influence tensors ${}^n\mathbb{T}^{\gamma\alpha}_{s,1}$ are stored into matrices of the form / $r=1$

$$T_{s_1|rs|ij|k} := ({}^nT^{\gamma\alpha}_{1,1})_{s_1rsijk}$$
$$T_{s_1s_2|rs|ij|k} := ({}^nT^{\gamma\alpha}_{2,1})_{s_1s_2rsijk}$$
$$T_{s_1s_2s_3|rs|ij|k} := ({}^nT^{\gamma\alpha}_{3,1})_{s_1s_2s_3rsijk}$$
$$T_{s_1s_2s_3s_4|rs|ij|k} := ({}^nT^{\gamma\alpha}_{4,1})_{s_1s_2s_3s_4rsijk}$$

$$\underset{6\times 6}{\left[{}^n\mathbb{T}^{\gamma\alpha}_{1,1}\right]} := \begin{bmatrix} T_{1|11|11|1} & T_{1|22|11|1} & \sqrt{2}T_{1|12|11|1} & T_{2|11|11|1} & T_{2|22|11|1} & \sqrt{2}T_{2|12|11|1} \\ T_{1|11|22|1} & T_{1|22|22|1} & \sqrt{2}T_{1|12|22|1} & T_{2|11|22|1} & T_{2|22|22|1} & \sqrt{2}T_{2|12|22|1} \\ \sqrt{2}T_{1|11|12|1} & \sqrt{2}T_{1|22|12|1} & \sqrt{4}T_{1|12|12|1} & \sqrt{2}T_{2|11|12|1} & \sqrt{2}T_{2|22|12|1} & \sqrt{4}T_{2|12|12|1} \\ T_{1|11|11|2} & T_{1|22|11|2} & \sqrt{2}T_{1|12|11|2} & T_{2|11|11|2} & T_{2|22|11|2} & \sqrt{2}T_{2|12|11|2} \\ T_{1|11|22|2} & T_{1|22|22|2} & \sqrt{2}T_{1|12|22|2} & T_{2|11|22|2} & T_{2|22|22|2} & \sqrt{2}T_{2|12|22|2} \\ \sqrt{2}T_{1|11|12|2} & \sqrt{2}T_{1|22|12|2} & \sqrt{4}T_{1|12|12|2} & \sqrt{2}T_{2|11|12|2} & \sqrt{2}T_{2|22|12|1} & \sqrt{4}T_{2|12|12|2} \end{bmatrix}$$

$$\underset{6\times 9}{\left[{}^n\mathbb{T}^{\gamma\alpha}_{2,1}\right]} := \begin{bmatrix} T_{11|11|11|1} & T_{11|22|11|1} & \sqrt{2}T_{11|12|11|1} & T_{22|11|11|1} & T_{22|22|11|1} & \sqrt{2}T_{22|12|11|1} & \sqrt{2}T_{12|11|11|1} & \sqrt{2}T_{12|22|11|1} & \sqrt{4}T_{12|12|11|1} \\ T_{11|11|22|1} & T_{11|22|22|1} & \sqrt{2}T_{11|12|22|1} & T_{22|11|22|1} & T_{22|22|22|1} & \sqrt{2}T_{22|12|22|1} & \sqrt{2}T_{12|11|22|1} & \sqrt{2}T_{12|22|22|1} & \sqrt{4}T_{12|12|22|1} \\ \sqrt{2}T_{11|11|12|1} & \sqrt{2}T_{11|22|12|1} & \sqrt{4}T_{11|12|12|1} & \sqrt{2}T_{22|11|12|1} & \sqrt{2}T_{22|22|12|1} & \sqrt{4}T_{22|12|12|1} & \sqrt{4}T_{12|11|12|1} & \sqrt{4}T_{12|22|12|1} & \sqrt{8}T_{12|12|12|1} \\ T_{11|11|11|2} & T_{11|22|11|1} & \sqrt{2}T_{11|12|11|2} & T_{22|11|11|2} & T_{22|22|11|2} & \sqrt{2}T_{22|12|11|2} & \sqrt{2}T_{12|11|11|2} & \sqrt{2}T_{12|22|11|2} & \sqrt{4}T_{12|12|11|2} \\ T_{11|11|22|2} & T_{11|22|22|1} & \sqrt{2}T_{11|12|22|2} & T_{22|11|22|2} & T_{22|22|22|2} & \sqrt{2}T_{22|12|22|2} & \sqrt{2}T_{12|11|22|2} & \sqrt{2}T_{12|22|22|2} & \sqrt{4}T_{12|12|22|2} \\ \sqrt{2}T_{11|11|12|2} & \sqrt{2}T_{11|22|12|2} & \sqrt{4}T_{11|12|12|2} & \sqrt{2}T_{22|11|12|2} & \sqrt{2}T_{22|22|12|2} & \sqrt{4}T_{22|12|12|2} & \sqrt{4}T_{12|11|12|2} & \sqrt{4}T_{12|22|12|2} & \sqrt{8}T_{12|12|12|2} \end{bmatrix}$$

```
T_local(γ , α ,s=2,r=1,I=4,J=1) = T_sym_infl(γ , α ,ns1=2,ns2=0,ijkl=2222,nr1=1,nr2=0)
                                = T_sym_infl(α , γ ,nr1=1,nr2=0,ijkl=2222,ns1=2,ns2=0)
```

```
ki=I%3, i=(I-ki)/3, if i%2==0: nr1=r-i/2
                          else: nr2=r-(i-1)/2
kj=J%3, j=(J-kj)/3, if j%2==0: ns1=s-j/2
                          else: ns2=s-(j-1)/2
list_of_ijkl=[[1111,1122,1111],[2211,2222,2212],
                               [1211,1222,1212]]
ijkl=list-of_ijkl[ki][kj]
```

$$\underset{6\times 12}{\left[{}^n\mathbb{T}^{\gamma\alpha}_{3,1}\right]} := \begin{bmatrix} T_{111|11|11|1} & T_{111|22|11|1} & \sqrt{2}T_{111|12|11|1} & T_{222|11|11|1} & T_{222|22|11|1} & \sqrt{2}T_{222|12|11|1} & \sqrt{3}T_{112|11|11|1} & \sqrt{3}T_{112|22|11|1} & \sqrt{6}T_{112|12|11|1} & \sqrt{3}T_{122|11|11|1} & \sqrt{3}T_{122|22|11|1} & \sqrt{6}T_{122|12|11|1} \\ T_{111|11|22|1} & T_{111|22|22|1} & \sqrt{2}T_{111|12|22|1} & T_{222|11|22|1} & T_{222|22|22|1} & \sqrt{2}T_{222|12|22|1} & \sqrt{3}T_{112|11|22|1} & \sqrt{3}T_{112|22|22|1} & \sqrt{6}T_{112|12|22|1} & \sqrt{3}T_{122|11|22|1} & \sqrt{3}T_{122|22|22|1} & \sqrt{6}T_{122|12|22|1} \\ \sqrt{2}T_{111|11|12|1} & \sqrt{2}T_{111|22|12|1} & \sqrt{4}T_{111|12|12|1} & \sqrt{2}T_{222|11|12|1} & \sqrt{2}T_{222|22|12|1} & \sqrt{4}T_{222|12|12|1} & \sqrt{6}T_{112|11|12|1} & \sqrt{6}T_{112|22|12|1} & \sqrt{12}T_{112|12|12|1} & \sqrt{6}T_{122|11|12|1} & \sqrt{6}T_{122|22|12|1} & \sqrt{12}T_{122|12|12|1} \\ T_{111|11|11|2} & T_{111|22|11|2} & \sqrt{2}T_{111|12|11|2} & T_{222|11|11|2} & T_{222|22|11|2} & \sqrt{2}T_{222|12|11|2} & \sqrt{3}T_{112|11|11|2} & \sqrt{3}T_{112|22|11|2} & \sqrt{6}T_{112|12|11|2} & \sqrt{3}T_{122|11|11|2} & \sqrt{3}T_{122|22|11|2} & \sqrt{6}T_{122|12|11|2} \\ T_{111|11|22|2} & T_{111|22|22|2} & \sqrt{2}T_{111|12|22|2} & T_{222|11|22|2} & T_{222|22|22|2} & \sqrt{2}T_{222|12|22|2} & \sqrt{3}T_{112|11|22|2} & \sqrt{3}T_{112|22|22|2} & \sqrt{6}T_{112|12|22|2} & \sqrt{3}T_{122|11|22|2} & \sqrt{3}T_{122|22|22|2} & \sqrt{6}T_{122|12|22|2} \\ \sqrt{2}T_{111|11|12|2} & \sqrt{2}T_{111|22|12|2} & \sqrt{4}T_{111|12|12|2} & \sqrt{2}T_{222|11|12|2} & \sqrt{2}T_{222|22|12|2} & \sqrt{4}T_{222|12|12|2} & \sqrt{6}T_{112|11|12|2} & \sqrt{6}T_{112|22|12|2} & \sqrt{12}T_{112|12|12|2} & \sqrt{6}T_{122|11|12|2} & \sqrt{6}T_{122|22|12|2} & \sqrt{12}T_{122|12|12|2} \end{bmatrix}$$

$$\underset{6\times 15}{\left[{}^n\mathbb{T}^{\gamma\alpha}_{4,1}\right]} := \begin{bmatrix} T_{1111|11|11|1} & T_{1111|22|11|1} & \sqrt{2}T_{1111|12|11|1} & T_{2222|11|11|1} & T_{2222|22|11|1} & \sqrt{2}T_{2222|12|11|1} & \sqrt{4}T_{1112|11|11|1} & \sqrt{4}T_{1112|22|11|1} & \sqrt{8}T_{1112|12|11|1} & \sqrt{6}T_{1122|11|11|1} & \sqrt{6}T_{1122|22|11|1} & \sqrt{12}T_{1122|12|11|1} & \sqrt{4}T_{1222|11|11|1} & \sqrt{4}T_{1222|22|11|1} & \sqrt{8}T_{1222|12|11|1} \\ T_{1111|11|22|1} & T_{1111|22|22|1} & \sqrt{2}T_{1111|12|22|1} & T_{2222|11|22|1} & T_{2222|22|22|1} & \sqrt{2}T_{2222|12|22|1} & \sqrt{4}T_{1112|11|22|1} & \sqrt{4}T_{1112|22|22|1} & \sqrt{8}T_{1112|12|22|1} & \sqrt{6}T_{1122|11|22|1} & \sqrt{6}T_{1122|22|22|1} & \sqrt{12}T_{1122|12|22|1} & \sqrt{4}T_{1222|11|22|1} & \sqrt{4}T_{1222|22|22|1} & \sqrt{8}T_{1222|12|22|1} \\ \sqrt{2}T_{1111|11|12|1} & \sqrt{2}T_{1111|22|12|1} & \sqrt{4}T_{1111|12|12|1} & \sqrt{2}T_{2222|11|12|1} & \sqrt{2}T_{2222|22|12|1} & \sqrt{4}T_{2222|12|12|1} & \sqrt{8}T_{1112|11|12|1} & \sqrt{8}T_{1112|22|12|1} & \sqrt{16}T_{1112|12|12|1} & \sqrt{12}T_{1122|11|12|1} & \sqrt{12}T_{1122|22|12|1} & \sqrt{24}T_{1122|12|12|1} & \sqrt{8}T_{1222|11|12|1} & \sqrt{8}T_{1222|22|12|1} & \sqrt{16}T_{1222|12|12|1} \\ T_{1111|11|11|2} & T_{1111|22|11|2} & \sqrt{2}T_{1111|12|11|2} & T_{2222|11|11|2} & T_{2222|22|11|2} & \sqrt{2}T_{2222|12|11|2} & \sqrt{4}T_{1112|11|11|2} & \sqrt{4}T_{1112|22|11|2} & \sqrt{8}T_{1112|12|11|2} & \sqrt{6}T_{1122|11|11|2} & \sqrt{6}T_{1122|22|11|2} & \sqrt{12}T_{1122|12|11|2} & \sqrt{4}T_{1222|11|11|2} & \sqrt{4}T_{1222|22|11|2} & \sqrt{8}T_{1222|12|11|2} \\ T_{1111|11|22|2} & T_{1111|22|22|2} & \sqrt{2}T_{1111|12|22|2} & T_{2222|11|22|2} & T_{2222|22|22|2} & \sqrt{2}T_{2222|12|22|2} & \sqrt{4}T_{1112|11|22|2} & \sqrt{4}T_{1112|22|22|2} & \sqrt{8}T_{1112|12|22|2} & \sqrt{6}T_{1122|11|22|2} & \sqrt{6}T_{1122|22|22|2} & \sqrt{12}T_{1122|12|22|2} & \sqrt{4}T_{1222|11|22|2} & \sqrt{4}T_{1222|22|22|2} & \sqrt{8}T_{1222|12|22|2} \\ \sqrt{2}T_{1111|11|12|2} & \sqrt{2}T_{1111|22|12|2} & \sqrt{4}T_{1111|12|12|2} & \sqrt{2}T_{2222|11|12|2} & \sqrt{2}T_{2222|22|12|2} & \sqrt{4}T_{2222|12|12|2} & \sqrt{8}T_{1112|11|12|2} & \sqrt{8}T_{1112|22|12|2} & \sqrt{16}T_{1112|12|12|2} & \sqrt{12}T_{1122|11|12|2} & \sqrt{12}T_{1122|22|12|2} & \sqrt{24}T_{1122|12|22|1} & \sqrt{8}T_{1222|11|12|2} & \sqrt{8}T_{1222|22|12|2} & \sqrt{16}T_{1222|12|12|2} \end{bmatrix}$$

"Generalized Mandel representation" for assembly of a global system of stationarity equations

The components of the influence tensors ${}^{n}\mathbb{T}^{\gamma\alpha}_{s,2}$ are stored into matrices of the form / $r = 2$

$$\underset{9\times 6}{\left[{}^{n}\mathbb{T}^{\gamma\alpha}_{1,2}\right]} := \begin{bmatrix}
T_{1|11|11|11} & T_{1|22|11|11} & \sqrt{2}T_{1|12|11|11} & T_{2|11|11|11} & T_{2|22|11|11} & \sqrt{2}T_{2|12|11|11} \\
T_{1|11|22|11} & T_{1|22|22|11} & \sqrt{2}T_{1|12|22|11} & T_{2|11|22|11} & T_{2|22|22|11} & \sqrt{2}T_{2|12|22|11} \\
\sqrt{2}T_{1|11|12|11} & \sqrt{2}T_{1|22|12|11} & \sqrt{4}T_{1|12|12|11} & \sqrt{2}T_{2|11|12|11} & \sqrt{2}T_{2|22|12|11} & \sqrt{4}T_{2|12|12|11} \\
T_{1|11|11|22} & T_{1|22|11|22} & \sqrt{2}T_{1|12|11|22} & T_{2|11|11|22} & T_{2|22|11|22} & \sqrt{2}T_{2|12|11|22} \\
T_{1|11|22|22} & T_{1|22|22|22} & \sqrt{2}T_{1|12|22|22} & T_{2|11|22|22} & T_{2|22|22|22} & \sqrt{2}T_{2|12|22|22} \\
\sqrt{2}T_{1|11|12|22} & \sqrt{2}T_{1|22|12|22} & \sqrt{4}T_{1|12|12|22} & \sqrt{2}T_{2|11|12|22} & \sqrt{2}T_{2|22|12|22} & \sqrt{4}T_{2|12|12|22} \\
\sqrt{2}T_{1|11|11|12} & \sqrt{2}T_{1|22|11|12} & \sqrt{4}T_{1|12|11|12} & \sqrt{2}T_{2|11|11|12} & \sqrt{2}T_{2|22|11|12} & \sqrt{4}T_{2|12|11|12} \\
\sqrt{2}T_{1|11|22|12} & \sqrt{2}T_{1|22|22|12} & \sqrt{4}T_{1|12|22|12} & \sqrt{2}T_{2|11|22|12} & \sqrt{2}T_{2|22|22|12} & \sqrt{4}T_{2|12|22|12} \\
\sqrt{4}T_{1|11|12|12} & \sqrt{4}T_{1|22|12|12} & \sqrt{8}T_{1|12|12|12} & \sqrt{4}T_{2|11|12|12} & \sqrt{4}T_{2|22|12|12} & \sqrt{8}T_{2|12|12|12}
\end{bmatrix}$$

$$T_{s_1|rs|ij|kl} := \left({}^{n}T^{\gamma\alpha}_{1,2}\right)_{s_1 rsijkl}$$

$$T_{s_1 s_2|rs|ij|kl} := \left({}^{n}T^{\gamma\alpha}_{2,2}\right)_{s_1 s_2 rsijkl}$$

$$T_{s_1 s_2 s_3|rs|ij|kl} := \left({}^{n}T^{\gamma\alpha}_{3,2}\right)_{s_1 s_2 s_3 rsijkl}$$

$$T_{s_1 s_2 s_3 s_4|rs|ij|kl} := \left({}^{n}T^{\gamma\alpha}_{4,2}\right)_{s_1 s_2 s_3 s_4 rsijkl}$$

$$\underset{9\times 9}{\left[{}^{n}\mathbb{T}^{\gamma\alpha}_{2,2}\right]} := \begin{bmatrix}
T_{11|11|11|11} & T_{11|22|11|11} & \sqrt{2}T_{11|12|11|11} & T_{22|11|11|11} & T_{22|22|11|11} & \sqrt{2}T_{22|12|11|11} & \sqrt{2}T_{12|11|11|11} & \sqrt{2}T_{12|22|11|11} & \sqrt{4}T_{12|12|11|11} \\
T_{11|11|22|11} & T_{11|22|22|11} & \sqrt{2}T_{11|12|22|11} & T_{22|11|22|11} & T_{22|22|22|11} & \sqrt{2}T_{22|12|22|11} & \sqrt{2}T_{12|11|22|11} & \sqrt{2}T_{12|22|22|11} & \sqrt{4}T_{12|12|22|11} \\
\sqrt{2}T_{11|11|12|11} & \sqrt{2}T_{11|22|12|11} & \sqrt{4}T_{11|12|12|11} & \sqrt{2}T_{22|11|12|11} & \sqrt{2}T_{22|22|12|11} & \sqrt{4}T_{22|12|12|11} & \sqrt{4}T_{12|11|12|11} & \sqrt{4}T_{12|22|12|11} & \sqrt{8}T_{12|12|12|11} \\
T_{11|11|11|22} & T_{11|22|11|22} & \sqrt{2}T_{11|12|11|22} & T_{22|11|11|22} & T_{22|22|11|22} & \sqrt{2}T_{22|12|11|22} & \sqrt{2}T_{12|11|11|22} & \sqrt{2}T_{12|22|11|22} & \sqrt{4}T_{12|12|11|22} \\
T_{11|11|22|22} & T_{11|22|22|22} & \sqrt{2}T_{11|12|22|22} & T_{22|11|22|22} & T_{22|22|22|22} & \sqrt{2}T_{22|12|22|22} & \sqrt{2}T_{12|11|22|22} & \sqrt{2}T_{12|22|22|22} & \sqrt{4}T_{12|12|22|22} \\
\sqrt{2}T_{11|11|12|22} & \sqrt{2}T_{11|22|12|22} & \sqrt{4}T_{11|12|12|22} & \sqrt{2}T_{22|11|12|22} & \sqrt{2}T_{22|22|12|22} & \sqrt{4}T_{22|12|12|22} & \sqrt{4}T_{12|11|12|22} & \sqrt{4}T_{12|22|12|22} & \sqrt{8}T_{12|12|12|22} \\
\sqrt{2}T_{11|11|11|12} & \sqrt{2}T_{11|22|11|12} & \sqrt{4}T_{11|12|11|12} & \sqrt{2}T_{22|11|11|12} & \sqrt{2}T_{22|22|11|12} & \sqrt{4}T_{22|12|11|12} & \sqrt{4}T_{12|11|11|12} & \sqrt{4}T_{12|22|11|12} & \sqrt{8}T_{12|12|11|12} \\
\sqrt{2}T_{11|11|22|12} & \sqrt{2}T_{11|22|22|12} & \sqrt{4}T_{11|12|22|12} & \sqrt{2}T_{22|11|22|12} & \sqrt{2}T_{22|22|22|12} & \sqrt{4}T_{22|12|22|12} & \sqrt{4}T_{12|11|22|12} & \sqrt{4}T_{12|22|22|12} & \sqrt{8}T_{12|12|22|12} \\
\sqrt{4}T_{11|11|12|12} & \sqrt{4}T_{11|22|12|12} & \sqrt{8}T_{11|12|12|12} & \sqrt{4}T_{22|11|12|12} & \sqrt{4}T_{22|22|12|12} & \sqrt{8}T_{22|12|12|12} & \sqrt{8}T_{12|11|12|12} & \sqrt{8}T_{12|22|12|12} & \sqrt{16}T_{12|12|12|12}
\end{bmatrix}$$

$$\underset{9\times 12}{\left[{}^{n}\mathbb{T}^{\gamma\alpha}_{3,2}\right]} := \begin{bmatrix}
T_{111|11|11|11} & T_{111|22|11|11} & \sqrt{2}T_{111|12|11|11} & T_{222|11|11|11} & T_{222|22|11|11} & \sqrt{2}T_{222|12|11|11} & \sqrt{3}T_{112|11|11|11} & \sqrt{3}T_{112|22|11|11} & \sqrt{6}T_{112|12|11|11} & \sqrt{3}T_{122|11|11|11} & \sqrt{3}T_{122|22|11|11} & \sqrt{6}T_{122|12|11|11} \\
T_{111|11|22|11} & T_{111|22|22|11} & \sqrt{2}T_{111|12|22|11} & T_{222|11|22|11} & T_{222|22|22|11} & \sqrt{2}T_{222|12|22|11} & \sqrt{3}T_{112|11|22|11} & \sqrt{3}T_{112|22|22|11} & \sqrt{6}T_{112|12|22|11} & \sqrt{3}T_{122|11|22|11} & \sqrt{3}T_{122|22|22|11} & \sqrt{6}T_{122|12|22|11} \\
\sqrt{2}T_{111|11|12|11} & \sqrt{2}T_{111|22|12|11} & \sqrt{8}T_{111|12|12|11} & \sqrt{2}T_{222|11|12|11} & \sqrt{2}T_{222|22|12|11} & \sqrt{8}T_{222|12|12|11} & \sqrt{6}T_{112|11|12|11} & \sqrt{6}T_{112|22|12|11} & \sqrt{12}T_{112|12|12|11} & \sqrt{6}T_{122|11|12|11} & \sqrt{6}T_{122|22|12|11} & \sqrt{12}T_{122|12|12|11} \\
T_{111|11|11|22} & T_{111|22|11|22} & \sqrt{2}T_{111|12|11|22} & T_{222|11|11|22} & T_{222|22|11|22} & \sqrt{2}T_{222|12|11|22} & \sqrt{3}T_{112|11|11|22} & \sqrt{3}T_{112|22|11|22} & \sqrt{6}T_{112|12|11|22} & \sqrt{3}T_{122|11|11|22} & \sqrt{3}T_{122|22|11|22} & \sqrt{6}T_{122|12|11|22} \\
T_{111|11|22|22} & T_{111|22|22|22} & \sqrt{2}T_{111|12|22|22} & T_{222|11|22|22} & T_{222|22|22|22} & \sqrt{2}T_{222|12|22|22} & \sqrt{3}T_{112|11|22|22} & \sqrt{3}T_{112|22|22|22} & \sqrt{6}T_{112|12|22|22} & \sqrt{3}T_{122|11|22|22} & \sqrt{3}T_{122|22|22|22} & \sqrt{6}T_{122|12|22|22} \\
\sqrt{2}T_{111|11|12|22} & \sqrt{2}T_{111|22|12|22} & \sqrt{8}T_{111|12|12|22} & \sqrt{2}T_{222|11|12|22} & \sqrt{2}T_{222|22|12|22} & \sqrt{8}T_{222|12|12|22} & \sqrt{6}T_{112|11|12|22} & \sqrt{6}T_{112|22|12|22} & \sqrt{12}T_{112|12|12|22} & \sqrt{6}T_{122|11|12|22} & \sqrt{6}T_{122|22|12|22} & \sqrt{12}T_{122|12|12|22} \\
\sqrt{2}T_{111|11|11|12} & \sqrt{2}T_{111|22|11|12} & \sqrt{4}T_{111|12|11|12} & \sqrt{2}T_{222|11|11|12} & \sqrt{2}T_{222|22|11|12} & \sqrt{4}T_{222|12|11|12} & \sqrt{6}T_{112|11|11|12} & \sqrt{6}T_{112|22|11|12} & \sqrt{12}T_{112|12|11|12} & \sqrt{6}T_{122|11|11|12} & \sqrt{6}T_{122|22|11|12} & \sqrt{12}T_{122|12|11|12} \\
\sqrt{2}T_{111|11|22|12} & \sqrt{2}T_{111|22|22|12} & \sqrt{4}T_{111|12|22|12} & \sqrt{2}T_{222|11|22|12} & \sqrt{2}T_{222|22|22|12} & \sqrt{4}T_{222|12|22|12} & \sqrt{6}T_{112|11|22|12} & \sqrt{6}T_{112|22|22|12} & \sqrt{12}T_{112|12|22|12} & \sqrt{6}T_{122|11|22|12} & \sqrt{6}T_{122|22|22|12} & \sqrt{12}T_{122|12|22|12} \\
\sqrt{4}T_{111|11|12|12} & \sqrt{4}T_{111|22|12|12} & \sqrt{8}T_{111|12|12|12} & \sqrt{4}T_{222|11|12|12} & \sqrt{4}T_{222|22|12|12} & \sqrt{8}T_{222|12|12|12} & \sqrt{12}T_{112|11|12|12} & \sqrt{12}T_{112|22|12|12} & \sqrt{24}T_{112|12|12|12} & \sqrt{12}T_{122|11|12|12} & \sqrt{12}T_{122|22|12|12} & \sqrt{24}T_{122|12|12|12}
\end{bmatrix}$$

$$\underset{9\times 15}{\left[{}^{n}\mathbb{T}^{\gamma\alpha}_{4,2}\right]} := \begin{bmatrix}
T_{1111|11|11|11} & T_{1111|22|11|11} & \sqrt{2}T_{1111|12|11|11} & T_{2222|11|11|11} & T_{2222|22|11|11} & \sqrt{2}T_{2222|12|11|11} & \sqrt{4}T_{1112|11|11|11} & \sqrt{4}T_{1112|22|11|11} & \sqrt{8}T_{1112|12|11|11} & \sqrt{6}T_{1122|11|11|11} & \sqrt{6}T_{1122|22|11|11} & \sqrt{12}T_{1122|12|11|11} & \sqrt{4}T_{1222|11|11|11} & \sqrt{4}T_{1222|22|11|11} & \sqrt{8}T_{1222|12|11|11} \\
T_{1111|11|22|11} & T_{1111|22|22|11} & \sqrt{2}T_{1111|12|22|11} & T_{2222|11|22|11} & T_{2222|22|22|11} & \sqrt{2}T_{2222|12|22|11} & \sqrt{4}T_{1112|11|22|11} & \sqrt{4}T_{1112|22|22|11} & \sqrt{8}T_{1112|12|22|11} & \sqrt{6}T_{1122|11|22|11} & \sqrt{6}T_{1122|22|22|11} & \sqrt{12}T_{1122|12|22|11} & \sqrt{4}T_{1222|11|22|11} & \sqrt{4}T_{1222|22|22|11} & \sqrt{8}T_{1222|12|22|11} \\
\sqrt{2}T_{1111|11|12|11} & \sqrt{2}T_{1111|22|12|11} & \sqrt{4}T_{1111|12|12|11} & \sqrt{2}T_{2222|11|12|11} & \sqrt{2}T_{2222|22|12|11} & \sqrt{4}T_{2222|12|12|11} & \sqrt{8}T_{1112|11|12|11} & \sqrt{8}T_{1112|22|12|11} & \sqrt{16}T_{1112|12|12|11} & \sqrt{12}T_{1122|11|12|11} & \sqrt{12}T_{1122|22|12|11} & \sqrt{24}T_{1122|12|12|11} & \sqrt{8}T_{1222|11|12|11} & \sqrt{8}T_{1222|22|12|11} & \sqrt{16}T_{1222|12|12|11} \\
T_{1111|11|11|22} & T_{1111|22|11|22} & \sqrt{2}T_{1111|12|11|22} & T_{2222|11|11|22} & T_{2222|22|11|22} & \sqrt{2}T_{2222|12|11|22} & \sqrt{4}T_{1112|11|11|22} & \sqrt{4}T_{1112|22|11|22} & \sqrt{8}T_{1112|12|11|22} & \sqrt{6}T_{1122|11|11|22} & \sqrt{6}T_{1122|22|11|22} & \sqrt{12}T_{1122|12|11|22} & \sqrt{4}T_{1222|11|11|22} & \sqrt{4}T_{1222|22|11|22} & \sqrt{8}T_{1222|12|11|22} \\
T_{1111|11|22|22} & T_{1111|22|22|22} & \sqrt{2}T_{1111|12|22|22} & T_{2222|11|22|22} & T_{2222|22|22|22} & \sqrt{2}T_{2222|12|22|22} & \sqrt{4}T_{1112|11|22|22} & \sqrt{4}T_{1112|22|22|22} & \sqrt{8}T_{1112|12|22|22} & \sqrt{6}T_{1122|11|22|22} & \sqrt{6}T_{1122|22|22|22} & \sqrt{12}T_{1122|12|22|22} & \sqrt{4}T_{1222|11|22|22} & \sqrt{4}T_{1222|22|22|22} & \sqrt{8}T_{1222|12|22|22} \\
\sqrt{2}T_{1111|11|12|22} & \sqrt{2}T_{1111|22|12|22} & \sqrt{4}T_{1111|12|12|22} & \sqrt{2}T_{2222|11|12|22} & \sqrt{2}T_{2222|22|12|22} & \sqrt{4}T_{2222|12|12|22} & \sqrt{8}T_{1112|11|12|22} & \sqrt{8}T_{1112|22|12|22} & \sqrt{16}T_{1112|12|12|22} & \sqrt{12}T_{1122|11|12|22} & \sqrt{12}T_{1122|22|12|22} & \sqrt{24}T_{1122|12|12|22} & \sqrt{8}T_{1222|11|12|22} & \sqrt{8}T_{1222|22|12|22} & \sqrt{16}T_{1222|12|12|22} \\
\sqrt{2}T_{1111|11|11|12} & \sqrt{2}T_{1111|22|11|12} & \sqrt{4}T_{1111|12|11|12} & \sqrt{2}T_{2222|11|11|12} & \sqrt{2}T_{2222|22|11|12} & \sqrt{4}T_{2222|12|11|12} & \sqrt{8}T_{1112|11|11|12} & \sqrt{8}T_{1112|22|11|12} & \sqrt{16}T_{1112|12|11|12} & \sqrt{12}T_{1122|11|11|12} & \sqrt{12}T_{1122|22|11|12} & \sqrt{24}T_{1122|12|11|12} & \sqrt{8}T_{1222|11|11|12} & \sqrt{8}T_{1222|22|11|12} & \sqrt{16}T_{1222|12|11|12} \\
\sqrt{2}T_{1111|11|22|12} & \sqrt{2}T_{1111|22|22|12} & \sqrt{4}T_{1111|12|22|12} & \sqrt{2}T_{2222|11|22|12} & \sqrt{2}T_{2222|22|22|12} & \sqrt{4}T_{2222|12|22|12} & \sqrt{8}T_{1112|11|22|12} & \sqrt{8}T_{1112|22|22|12} & \sqrt{16}T_{1112|12|22|12} & \sqrt{12}T_{1122|11|22|12} & \sqrt{12}T_{1122|22|22|12} & \sqrt{24}T_{1122|12|22|12} & \sqrt{8}T_{1222|11|22|12} & \sqrt{8}T_{1222|22|22|12} & \sqrt{16}T_{1222|12|22|12} \\
\sqrt{4}T_{1111|11|12|12} & \sqrt{4}T_{1111|22|12|12} & \sqrt{8}T_{1111|12|12|12} & \sqrt{4}T_{2222|11|12|12} & \sqrt{4}T_{2222|22|12|12} & \sqrt{8}T_{2222|12|12|12} & \sqrt{16}T_{1112|11|12|12} & \sqrt{16}T_{1112|22|12|12} & \sqrt{32}T_{1112|12|12|12} & \sqrt{24}T_{1122|11|12|12} & \sqrt{24}T_{1122|22|12|12} & \sqrt{48}T_{1122|12|12|12} & \sqrt{16}T_{1222|11|12|12} & \sqrt{16}T_{1222|22|12|12} & \sqrt{32}T_{1222|12|12|12}
\end{bmatrix}$$

“Generalized Mandel representation” for assembly of a global system of stationarity equations

The components of the influence tensors ${}^{n}\mathbb{T}^{\gamma\alpha}_{s,3}$ are stored into matrices of the form / $r = 3$

$$T_{s_1|rs|ij|klm} := ({}^{n}T^{\gamma\alpha}_{1,3})_{s_1 rsijklm}$$

$$T_{s_1 s_2|rs|ij|klm} := ({}^{n}T^{\gamma\alpha}_{2,3})_{s_1 s_2 rsijklm}$$

$$\underset{12\times 6}{[{}^{n}\mathbb{T}^{\gamma\alpha}_{1,3}]} := \begin{bmatrix}
T_{1|11|11|111} & T_{1|22|11|111} & \sqrt{2}T_{1|12|11|111} & T_{2|11|11|111} & T_{2|22|11|111} & \sqrt{2}T_{2|12|11|111} \\
T_{1|11|22|111} & T_{1|22|22|111} & \sqrt{2}T_{1|12|22|111} & T_{2|11|22|111} & T_{2|22|22|111} & \sqrt{2}T_{2|12|22|111} \\
\sqrt{2}T_{1|11|12|111} & \sqrt{2}T_{1|22|12|111} & \sqrt{4}T_{1|12|12|111} & \sqrt{2}T_{2|11|12|111} & \sqrt{2}T_{2|22|12|111} & \sqrt{4}T_{2|12|12|111} \\
T_{1|11|11|222} & T_{1|22|11|222} & \sqrt{2}T_{1|12|11|222} & T_{2|11|11|222} & T_{2|22|11|222} & \sqrt{2}T_{2|12|11|222} \\
T_{1|11|22|222} & T_{1|22|22|222} & \sqrt{2}T_{1|12|22|222} & T_{2|11|22|222} & T_{2|22|22|222} & \sqrt{2}T_{2|12|22|222} \\
\sqrt{2}T_{1|11|12|222} & \sqrt{2}T_{1|22|12|222} & \sqrt{4}T_{1|12|12|222} & \sqrt{2}T_{2|11|12|222} & \sqrt{2}T_{2|22|12|222} & \sqrt{4}T_{2|12|12|222} \\
\sqrt{3}T_{1|11|11|112} & \sqrt{3}T_{1|22|11|112} & \sqrt{6}T_{1|12|11|112} & \sqrt{3}T_{2|11|11|112} & \sqrt{3}T_{2|22|11|112} & \sqrt{6}T_{2|12|11|112} \\
\sqrt{3}T_{1|11|22|112} & \sqrt{3}T_{1|22|22|112} & \sqrt{6}T_{1|12|22|112} & \sqrt{3}T_{2|11|22|112} & \sqrt{3}T_{2|22|22|112} & \sqrt{6}T_{2|12|22|112} \\
\sqrt{6}T_{1|11|12|112} & \sqrt{6}T_{1|22|12|112} & \sqrt{12}T_{1|12|12|112} & \sqrt{6}T_{2|11|12|112} & \sqrt{6}T_{2|22|12|112} & \sqrt{12}T_{2|12|12|112} \\
\sqrt{3}T_{1|11|11|122} & \sqrt{3}T_{1|22|11|122} & \sqrt{6}T_{1|12|11|122} & \sqrt{3}T_{2|11|11|122} & \sqrt{3}T_{2|22|11|122} & \sqrt{6}T_{2|12|11|122} \\
\sqrt{3}T_{1|11|22|122} & \sqrt{3}T_{1|22|22|122} & \sqrt{6}T_{1|12|22|122} & \sqrt{3}T_{2|11|22|122} & \sqrt{3}T_{2|22|22|122} & \sqrt{6}T_{2|12|22|122} \\
\sqrt{6}T_{1|11|12|122} & \sqrt{6}T_{1|22|12|122} & \sqrt{12}T_{1|12|12|122} & \sqrt{6}T_{2|11|12|122} & \sqrt{6}T_{2|22|12|122} & \sqrt{12}T_{2|12|12|122}
\end{bmatrix}$$

$$\underset{12\times 9}{[{}^{n}\mathbb{T}^{\gamma\alpha}_{2,3}]} := \begin{bmatrix}
T_{11|11|11|111} & T_{11|22|11|111} & \sqrt{2}T_{11|12|11|111} & T_{22|11|11|111} & T_{22|22|11|111} & \sqrt{2}T_{22|12|11|111} & \sqrt{2}T_{12|11|11|111} & \sqrt{2}T_{12|22|11|111} & \sqrt{4}T_{12|12|11|111} \\
T_{11|11|22|111} & T_{11|22|22|111} & \sqrt{2}T_{11|12|22|111} & T_{22|11|22|111} & T_{22|22|22|111} & \sqrt{2}T_{22|12|22|111} & \sqrt{2}T_{12|11|22|111} & \sqrt{2}T_{12|22|22|111} & \sqrt{4}T_{12|12|22|111} \\
\sqrt{2}T_{11|11|12|111} & \sqrt{2}T_{11|22|12|111} & \sqrt{4}T_{11|12|12|111} & \sqrt{2}T_{22|11|12|111} & \sqrt{2}T_{22|22|12|111} & \sqrt{4}T_{22|12|12|111} & \sqrt{4}T_{12|11|12|111} & \sqrt{4}T_{12|22|12|111} & \sqrt{8}T_{12|12|12|111} \\
T_{11|11|11|222} & T_{11|22|11|222} & \sqrt{2}T_{11|12|11|222} & T_{22|11|11|222} & T_{22|22|11|222} & \sqrt{2}T_{22|12|11|222} & \sqrt{2}T_{12|11|11|222} & \sqrt{2}T_{12|22|11|222} & \sqrt{4}T_{12|12|11|222} \\
T_{11|11|22|222} & T_{11|22|22|222} & \sqrt{2}T_{11|12|22|222} & T_{22|11|22|222} & T_{22|22|22|222} & \sqrt{2}T_{22|12|22|222} & \sqrt{2}T_{12|11|22|222} & \sqrt{2}T_{12|22|22|222} & \sqrt{4}T_{12|12|22|222} \\
\sqrt{2}T_{11|11|12|222} & \sqrt{2}T_{11|22|12|222} & \sqrt{4}T_{11|12|12|222} & \sqrt{2}T_{22|11|12|222} & \sqrt{2}T_{22|22|12|222} & \sqrt{4}T_{22|12|12|222} & \sqrt{4}T_{12|11|12|222} & \sqrt{4}T_{12|22|12|222} & \sqrt{8}T_{12|12|12|222} \\
\sqrt{3}T_{11|11|11|112} & \sqrt{3}T_{11|22|11|112} & \sqrt{6}T_{11|12|11|112} & \sqrt{3}T_{22|11|11|112} & \sqrt{3}T_{22|22|11|112} & \sqrt{6}T_{22|12|11|112} & \sqrt{6}T_{12|11|11|112} & \sqrt{6}T_{12|22|11|112} & \sqrt{12}T_{12|12|11|112} \\
\sqrt{3}T_{11|11|22|112} & \sqrt{3}T_{11|22|22|112} & \sqrt{6}T_{11|12|22|112} & \sqrt{3}T_{22|11|22|112} & \sqrt{3}T_{22|22|22|112} & \sqrt{6}T_{22|12|22|112} & \sqrt{6}T_{12|11|22|112} & \sqrt{6}T_{12|22|22|112} & \sqrt{12}T_{12|12|22|112} \\
\sqrt{6}T_{11|11|12|112} & \sqrt{6}T_{11|22|12|112} & \sqrt{12}T_{11|12|12|112} & \sqrt{6}T_{22|11|12|112} & \sqrt{6}T_{22|22|12|112} & \sqrt{12}T_{22|12|12|112} & \sqrt{12}T_{12|11|12|112} & \sqrt{12}T_{12|22|12|112} & \sqrt{24}T_{12|12|12|112} \\
\sqrt{3}T_{11|11|11|122} & \sqrt{3}T_{11|22|11|122} & \sqrt{6}T_{11|12|11|122} & \sqrt{3}T_{22|11|11|122} & \sqrt{3}T_{22|22|11|122} & \sqrt{6}T_{22|12|11|122} & \sqrt{6}T_{12|11|11|122} & \sqrt{6}T_{12|22|11|122} & \sqrt{12}T_{12|12|11|122} \\
\sqrt{3}T_{11|11|22|122} & \sqrt{3}T_{11|22|22|122} & \sqrt{6}T_{11|12|22|122} & \sqrt{3}T_{22|11|22|122} & \sqrt{3}T_{22|22|22|122} & \sqrt{6}T_{22|12|22|122} & \sqrt{6}T_{12|11|22|122} & \sqrt{6}T_{12|22|22|122} & \sqrt{12}T_{12|12|22|122} \\
\sqrt{6}T_{11|11|12|122} & \sqrt{6}T_{11|22|12|122} & \sqrt{12}T_{11|12|12|122} & \sqrt{6}T_{22|11|12|122} & \sqrt{6}T_{22|22|12|122} & \sqrt{12}T_{22|12|12|122} & \sqrt{12}T_{12|11|12|122} & \sqrt{12}T_{12|22|12|122} & \sqrt{24}T_{12|12|12|122}
\end{bmatrix}$$

“Generalized Mandel representation” for assembly of a global system of stationarity equations

The components of the influence tensors ${}^{n}\mathbb{T}^{\gamma\alpha}_{s,3}$ are stored into matrices of the form / $r = 3$

$$T_{s_1 s_2 s_3|rs|ij|klm} := ({}^{n}T^{\gamma\alpha}_{3,3})_{s_1 s_2 s_3 rsijklm}$$

$$T_{s_1 s_2 s_3 s_4|rs|ij|klm} := ({}^{n}T^{\gamma\alpha}_{4,3})_{s_1 s_2 s_3 s_4 rsijklm}$$

$$\underset{12\times 12}{[{}^{n}\mathbb{T}^{\gamma\alpha}_{3,3}]} := \begin{bmatrix}
T_{111|11|11|111} & T_{111|22|11|111} & \sqrt{2}T_{111|12|11|111} & T_{222|11|11|111} & T_{222|22|11|111} & \sqrt{2}T_{222|12|11|111} & \sqrt{3}T_{112|11|11|111} & \sqrt{3}T_{112|22|11|111} & \sqrt{6}T_{112|12|11|111} & \sqrt{3}T_{122|11|11|111} & \sqrt{3}T_{122|22|11|111} & \sqrt{6}T_{122|12|11|111} \\
T_{111|11|22|111} & T_{111|22|22|111} & \sqrt{2}T_{111|12|22|111} & T_{222|11|22|111} & T_{222|22|22|111} & \sqrt{2}T_{222|12|22|111} & \sqrt{3}T_{112|11|22|111} & \sqrt{3}T_{112|22|22|111} & \sqrt{6}T_{112|12|22|111} & \sqrt{3}T_{122|11|22|111} & \sqrt{3}T_{122|22|22|111} & \sqrt{6}T_{122|12|22|111} \\
\sqrt{2}T_{111|11|12|111} & \sqrt{2}T_{111|22|12|111} & \sqrt{8}T_{111|12|12|111} & \sqrt{2}T_{222|11|12|111} & \sqrt{2}T_{222|22|12|111} & \sqrt{8}T_{222|12|12|111} & \sqrt{6}T_{112|11|12|111} & \sqrt{6}T_{112|22|12|111} & \sqrt{12}T_{112|12|12|111} & \sqrt{6}T_{122|11|12|111} & \sqrt{6}T_{122|22|12|111} & \sqrt{12}T_{122|12|12|111} \\
T_{111|11|11|222} & T_{111|22|11|222} & \sqrt{2}T_{111|12|11|222} & T_{222|11|11|222} & T_{222|22|11|222} & \sqrt{2}T_{222|12|11|222} & \sqrt{3}T_{112|11|11|222} & \sqrt{3}T_{112|22|11|222} & \sqrt{6}T_{112|12|11|222} & \sqrt{3}T_{122|11|11|222} & \sqrt{3}T_{122|22|11|222} & \sqrt{6}T_{122|12|11|222} \\
T_{111|11|22|222} & T_{111|22|22|222} & \sqrt{2}T_{111|12|22|222} & T_{222|11|22|222} & T_{222|22|22|222} & \sqrt{2}T_{222|12|22|222} & \sqrt{3}T_{112|11|22|222} & \sqrt{3}T_{112|22|22|222} & \sqrt{6}T_{112|12|22|222} & \sqrt{3}T_{122|11|22|222} & \sqrt{3}T_{122|22|22|222} & \sqrt{6}T_{122|12|22|222} \\
\sqrt{2}T_{111|11|12|222} & \sqrt{2}T_{111|22|12|222} & \sqrt{8}T_{111|12|12|222} & \sqrt{2}T_{222|11|12|222} & \sqrt{2}T_{222|22|12|222} & \sqrt{8}T_{222|12|12|222} & \sqrt{6}T_{112|11|12|222} & \sqrt{6}T_{112|22|12|222} & \sqrt{12}T_{112|12|12|222} & \sqrt{6}T_{122|11|12|22} & \sqrt{6}T_{122|22|12|222} & \sqrt{12}T_{122|12|12|222} \\
\sqrt{3}T_{111|11|11|112} & \sqrt{3}T_{111|22|11|112} & \sqrt{6}T_{111|12|11|112} & \sqrt{3}T_{222|11|11|112} & \sqrt{3}T_{222|22|11|112} & \sqrt{6}T_{222|12|11|112} & \sqrt{9}T_{112|11|11|112} & \sqrt{9}T_{112|22|11|112} & \sqrt{18}T_{112|12|11|112} & \sqrt{9}T_{122|11|11|112} & \sqrt{9}T_{122|22|11|112} & \sqrt{18}T_{122|12|11|112} \\
\sqrt{3}T_{111|11|22|112} & \sqrt{3}T_{111|22|22|112} & \sqrt{6}T_{111|12|22|112} & \sqrt{3}T_{222|11|22|112} & \sqrt{3}T_{222|22|22|112} & \sqrt{6}T_{222|12|22|112} & \sqrt{9}T_{112|11|22|112} & \sqrt{9}T_{112|22|22|112} & \sqrt{18}T_{112|12|22|112} & \sqrt{9}T_{122|11|22|112} & \sqrt{9}T_{122|22|22|112} & \sqrt{18}T_{122|12|22|112} \\
\sqrt{6}T_{111|11|12|112} & \sqrt{6}T_{111|22|12|112} & \sqrt{12}T_{111|12|12|112} & \sqrt{6}T_{222|11|12|112} & \sqrt{6}T_{222|22|12|112} & \sqrt{12}T_{222|12|12|112} & \sqrt{18}T_{112|11|12|112} & \sqrt{18}T_{112|22|12|112} & \sqrt{36}T_{112|12|12|112} & \sqrt{18}T_{122|11|12|112} & \sqrt{18}T_{122|22|12|112} & \sqrt{36}T_{122|12|12|112} \\
\sqrt{3}T_{111|11|11|122} & \sqrt{3}T_{111|22|11|122} & \sqrt{6}T_{111|12|11|122} & \sqrt{3}T_{222|11|11|122} & \sqrt{3}T_{222|22|11|122} & \sqrt{6}T_{222|12|11|122} & \sqrt{9}T_{112|11|11|122} & \sqrt{9}T_{112|22|11|122} & \sqrt{18}T_{112|12|11|122} & \sqrt{9}T_{122|11|11|122} & \sqrt{9}T_{122|22|11|122} & \sqrt{18}T_{122|12|11|122} \\
\sqrt{3}T_{111|11|22|122} & \sqrt{3}T_{111|22|22|122} & \sqrt{6}T_{111|12|22|122} & \sqrt{3}T_{222|11|22|122} & \sqrt{3}T_{222|22|22|122} & \sqrt{6}T_{222|12|22|122} & \sqrt{9}T_{112|11|22|122} & \sqrt{9}T_{112|22|22|122} & \sqrt{18}T_{112|12|22|122} & \sqrt{9}T_{122|11|22|122} & \sqrt{9}T_{122|22|22|122} & \sqrt{18}T_{122|12|22|122} \\
\sqrt{6}T_{111|11|12|122} & \sqrt{6}T_{111|22|12|122} & \sqrt{12}T_{111|12|12|122} & \sqrt{6}T_{222|11|12|122} & \sqrt{6}T_{222|22|12|122} & \sqrt{12}T_{222|12|12|122} & \sqrt{18}T_{112|11|12|122} & \sqrt{18}T_{112|22|12|122} & \sqrt{36}T_{112|12|12|122} & \sqrt{18}T_{122|11|12|122} & \sqrt{18}T_{122|22|12|122} & \sqrt{36}T_{122|12|12|122}
\end{bmatrix}$$

$$\underset{12\times 15}{[{}^{n}\mathbb{T}^{\gamma\alpha}_{4,3}]} := \begin{bmatrix}
T_{1111|11|11|111} & T_{1111|22|11|111} & \sqrt{2}T_{1111|12|11|111} & T_{2222|11|11|111} & T_{2222|22|11|111} & \sqrt{2}T_{2222|12|11|111} & \sqrt{4}T_{1112|11|11|111} & \sqrt{4}T_{1112|22|11|111} & \sqrt{8}T_{1112|12|11|111} & \sqrt{6}T_{1122|11|11|111} & \sqrt{6}T_{1122|22|11|111} & \sqrt{12}T_{1122|12|11|111} & \sqrt{4}T_{1222|11|11|111} & \sqrt{4}T_{1222|22|11|111} & \sqrt{8}T_{1222|12|11|111} \\
T_{1111|11|22|111} & T_{1111|22|22|111} & \sqrt{2}T_{1111|12|22|111} & T_{2222|11|22|111} & T_{2222|22|22|111} & \sqrt{2}T_{2222|12|22|111} & \sqrt{4}T_{1112|11|22|111} & \sqrt{4}T_{1112|22|22|111} & \sqrt{8}T_{1112|12|22|111} & \sqrt{6}T_{1122|11|22|111} & \sqrt{6}T_{1122|22|22|111} & \sqrt{12}T_{1122|12|22|111} & \sqrt{4}T_{1222|11|22|111} & \sqrt{4}T_{1222|22|22|111} & \sqrt{8}T_{1222|12|22|111} \\
\sqrt{2}T_{1111|11|12|111} & \sqrt{2}T_{1111|22|12|111} & \sqrt{4}T_{1111|12|12|111} & \sqrt{2}T_{2222|11|12|111} & \sqrt{2}T_{2222|22|12|111} & \sqrt{4}T_{2222|12|12|111} & \sqrt{8}T_{1112|11|12|111} & \sqrt{8}T_{1112|22|12|111} & \sqrt{16}T_{1112|12|12|111} & \sqrt{12}T_{1122|11|12|111} & \sqrt{12}T_{1122|22|12|111} & \sqrt{24}T_{1122|12|12|111} & \sqrt{8}T_{1222|11|12|111} & \sqrt{8}T_{1222|22|12|111} & \sqrt{16}T_{1222|12|12|111} \\
T_{1111|11|11|222} & T_{1111|22|11|222} & \sqrt{2}T_{1111|12|11|222} & T_{2222|11|11|222} & T_{2222|22|11|222} & \sqrt{2}T_{2222|12|11|222} & \sqrt{4}T_{1112|11|11|222} & \sqrt{4}T_{1112|22|11|222} & \sqrt{8}T_{1112|12|11|222} & \sqrt{6}T_{1122|11|11|222} & \sqrt{6}T_{1122|22|11|222} & \sqrt{12}T_{1122|12|11|222} & \sqrt{4}T_{1222|11|11|222} & \sqrt{4}T_{1222|22|11|222} & \sqrt{8}T_{1222|12|11|222} \\
T_{1111|11|22|222} & T_{1111|22|22|222} & \sqrt{2}T_{1111|12|22|222} & T_{2222|11|22|222} & T_{2222|22|22|222} & \sqrt{2}T_{2222|12|22|222} & \sqrt{4}T_{1112|11|22|222} & \sqrt{4}T_{1112|22|22|222} & \sqrt{8}T_{1112|12|22|222} & \sqrt{6}T_{1122|11|22|222} & \sqrt{6}T_{1122|22|22|222} & \sqrt{12}T_{1122|12|22|222} & \sqrt{4}T_{1222|11|22|222} & \sqrt{4}T_{1222|22|22|222} & \sqrt{8}T_{1222|12|22|222} \\
\sqrt{2}T_{1111|11|12|222} & \sqrt{2}T_{1111|22|12|222} & \sqrt{4}T_{1111|12|12|222} & \sqrt{2}T_{2222|11|12|222} & \sqrt{2}T_{2222|22|12|222} & \sqrt{4}T_{2222|12|12|222} & \sqrt{8}T_{1112|11|12|222} & \sqrt{8}T_{1112|22|12|222} & \sqrt{16}T_{1112|12|12|222} & \sqrt{12}T_{1122|11|12|222} & \sqrt{12}T_{1122|22|12|222} & \sqrt{24}T_{1122|12|12|222} & \sqrt{8}T_{1222|11|12|222} & \sqrt{8}T_{1222|22|12|222} & \sqrt{16}T_{1222|12|12|222} \\
\sqrt{3}T_{1111|11|11|112} & \sqrt{3}T_{1111|22|11|112} & \sqrt{6}T_{1111|12|11|112} & \sqrt{3}T_{2222|11|11|112} & \sqrt{3}T_{2222|22|11|112} & \sqrt{6}T_{2222|12|11|112} & \sqrt{12}T_{1112|11|11|112} & \sqrt{12}T_{1112|22|11|112} & \sqrt{24}T_{1112|12|11|112} & \sqrt{18}T_{1122|11|11|112} & \sqrt{18}T_{1122|22|11|112} & \sqrt{36}T_{1122|12|11|112} & \sqrt{12}T_{1222|11|11|112} & \sqrt{12}T_{1222|22|11|112} & \sqrt{24}T_{1222|12|11|112} \\
\sqrt{3}T_{1111|11|22|112} & \sqrt{3}T_{1111|22|22|112} & \sqrt{6}T_{1111|12|22|112} & \sqrt{3}T_{2222|11|22|112} & \sqrt{3}T_{2222|22|22|112} & \sqrt{6}T_{2222|12|22|112} & \sqrt{12}T_{1112|11|22|112} & \sqrt{12}T_{1112|22|22|112} & \sqrt{24}T_{1112|12|22|112} & \sqrt{18}T_{1122|11|22|112} & \sqrt{18}T_{1122|22|22|112} & \sqrt{36}T_{1122|12|22|112} & \sqrt{12}T_{1222|11|22|112} & \sqrt{12}T_{1222|22|22|112} & \sqrt{24}T_{1222|12|22|112} \\
\sqrt{6}T_{1111|11|12|112} & \sqrt{6}T_{1111|22|12|112} & \sqrt{12}T_{1111|12|12|112} & \sqrt{6}T_{2222|11|12|112} & \sqrt{6}T_{2222|22|12|112} & \sqrt{12}T_{2222|12|12|112} & \sqrt{24}T_{1112|11|12|112} & \sqrt{24}T_{1112|22|12|112} & \sqrt{48}T_{1112|12|12|112} & \sqrt{36}T_{1122|11|12|112} & \sqrt{36}T_{1122|22|12|112} & \sqrt{72}T_{1122|12|12|112} & \sqrt{24}T_{1222|11|12|112} & \sqrt{24}T_{1222|22|12|112} & \sqrt{48}T_{1222|12|12|112} \\
\sqrt{3}T_{1111|11|11|122} & \sqrt{3}T_{1111|22|11|122} & \sqrt{6}T_{1111|12|11|122} & \sqrt{3}T_{2222|11|11|122} & \sqrt{3}T_{2222|22|11|122} & \sqrt{6}T_{2222|12|11|122} & \sqrt{12}T_{1112|11|11|122} & \sqrt{12}T_{1112|22|11|122} & \sqrt{24}T_{1112|12|11|122} & \sqrt{18}T_{1122|11|11|122} & \sqrt{18}T_{1122|22|11|122} & \sqrt{36}T_{1122|12|11|122} & \sqrt{12}T_{1222|11|11|122} & \sqrt{12}T_{1222|22|11|122} & \sqrt{24}T_{1222|12|11|122} \\
\sqrt{3}T_{1111|11|22|122} & \sqrt{3}T_{1111|22|22|122} & \sqrt{6}T_{1111|12|22|122} & \sqrt{3}T_{2222|11|22|122} & \sqrt{3}T_{2222|22|22|122} & \sqrt{6}T_{2222|12|22|122} & \sqrt{12}T_{1112|11|22|122} & \sqrt{12}T_{1112|22|22|122} & \sqrt{24}T_{1112|12|22|122} & \sqrt{18}T_{1122|11|22|122} & \sqrt{18}T_{1122|22|22|122} & \sqrt{36}T_{1122|12|22|122} & \sqrt{12}T_{1222|11|22|122} & \sqrt{12}T_{1222|22|22|122} & \sqrt{24}T_{1222|12|22|122} \\
\sqrt{6}T_{1111|11|12|122} & \sqrt{6}T_{1111|22|12|122} & \sqrt{12}T_{1111|12|12|122} & \sqrt{6}T_{2222|11|12|122} & \sqrt{6}T_{2222|22|12|122} & \sqrt{12}T_{2222|12|12|122} & \sqrt{24}T_{1112|11|12|122} & \sqrt{24}T_{1112|22|12|122} & \sqrt{48}T_{1112|12|12|122} & \sqrt{36}T_{1122|11|12|122} & \sqrt{36}T_{1122|22|12|122} & \sqrt{72}T_{1122|12|12|122} & \sqrt{24}T_{1222|11|12|122} & \sqrt{24}T_{1222|22|12|122} & \sqrt{48}T_{1222|12|12|122}
\end{bmatrix}$$

"Generalized Mandel representation" for assembly of a global system of stationarity equations

The components of the influence tensors ${}^{n}\mathbb{T}^{\gamma\alpha}_{s,4}$ are stored into matrices of the form / $r = 4$

$$T_{s_1|rs|ij|klmn} := ({}^{n}T^{\gamma\alpha}_{1,4})_{s_1 rsijklmn}$$

$$T_{s_1 s_2|rs|ij|klmn} := ({}^{n}T^{\gamma\alpha}_{2,4})_{s_1 s_2 rsijklmn}$$

$$\underset{15\times 6}{[{}^{n}\mathbb{T}^{\gamma\alpha}_{1,4}]} := \begin{bmatrix}
T_{1|11|11|1111} & T_{1|22|11|1111} & \sqrt{2}T_{1|12|11|1111} & T_{2|11|11|1111} & T_{2|22|11|1111} & \sqrt{2}T_{2|12|11|1111} \\
T_{1|11|22|1111} & T_{1|22|22|1111} & \sqrt{2}T_{1|12|22|1111} & T_{2|11|22|1111} & T_{2|22|22|1111} & \sqrt{2}T_{2|12|22|1111} \\
\sqrt{2}T_{1|11|12|1111} & \sqrt{2}T_{1|22|12|1111} & \sqrt{4}T_{1|12|12|1111} & \sqrt{2}T_{2|11|12|1111} & \sqrt{2}T_{2|22|12|1111} & \sqrt{4}T_{2|12|12|1111} \\
T_{1|11|11|2222} & T_{1|22|11|2222} & \sqrt{2}T_{1|12|11|2222} & T_{2|11|11|2222} & T_{2|22|11|2222} & \sqrt{2}T_{2|12|11|2222} \\
T_{1|11|22|2222} & T_{1|22|22|2222} & \sqrt{2}T_{1|12|22|2222} & T_{2|11|22|2222} & T_{2|22|22|2222} & \sqrt{2}T_{2|12|22|2222} \\
\sqrt{2}T_{1|11|12|2222} & \sqrt{2}T_{1|22|12|2222} & \sqrt{4}T_{1|12|12|2222} & \sqrt{2}T_{2|11|12|2222} & \sqrt{2}T_{2|22|12|2222} & \sqrt{4}T_{2|12|12|2222} \\
\sqrt{4}T_{1|11|11|1112} & \sqrt{4}T_{1|22|11|1112} & \sqrt{8}T_{1|12|11|1112} & \sqrt{4}T_{2|11|11|1112} & \sqrt{4}T_{2|22|11|1112} & \sqrt{8}T_{2|12|11|1112} \\
\sqrt{4}T_{1|11|22|1112} & \sqrt{4}T_{1|22|22|1112} & \sqrt{8}T_{1|12|22|1112} & \sqrt{4}T_{2|11|22|1112} & \sqrt{4}T_{2|22|22|1112} & \sqrt{8}T_{2|12|22|1112} \\
\sqrt{8}T_{1|11|12|1112} & \sqrt{8}T_{1|22|12|1112} & \sqrt{16}T_{1|12|12|1112} & \sqrt{8}T_{2|11|12|1112} & \sqrt{8}T_{2|22|12|1112} & \sqrt{16}T_{2|12|12|1112} \\
\sqrt{6}T_{1|11|11|1122} & \sqrt{6}T_{1|22|11|1122} & \sqrt{12}T_{1|12|11|1122} & \sqrt{6}T_{2|11|11|1122} & \sqrt{6}T_{2|22|11|1122} & \sqrt{12}T_{2|12|11|1122} \\
\sqrt{6}T_{1|11|22|1122} & \sqrt{6}T_{1|22|22|1122} & \sqrt{12}T_{1|12|22|1122} & \sqrt{6}T_{2|11|22|1122} & \sqrt{6}T_{2|22|22|1122} & \sqrt{12}T_{2|12|22|1122} \\
\sqrt{12}T_{1|11|12|1122} & \sqrt{12}T_{1|22|12|1122} & \sqrt{24}T_{1|12|12|1122} & \sqrt{12}T_{2|11|12|1122} & \sqrt{12}T_{2|22|12|1122} & \sqrt{24}T_{2|12|12|1122} \\
\sqrt{4}T_{1|11|11|1222} & \sqrt{4}T_{1|22|11|1222} & \sqrt{8}T_{1|12|11|1222} & \sqrt{4}T_{2|11|11|1222} & \sqrt{4}T_{2|22|11|1222} & \sqrt{8}T_{2|12|11|1222} \\
\sqrt{4}T_{1|11|22|1222} & \sqrt{4}T_{1|22|22|1222} & \sqrt{8}T_{1|12|22|1222} & \sqrt{4}T_{2|11|22|1222} & \sqrt{4}T_{2|22|22|1222} & \sqrt{8}T_{2|12|22|1222} \\
\sqrt{8}T_{1|11|12|1222} & \sqrt{8}T_{1|22|12|1222} & \sqrt{16}T_{1|12|12|1222} & \sqrt{8}T_{2|11|12|1222} & \sqrt{8}T_{2|22|12|1222} & \sqrt{16}T_{2|12|12|1222}
\end{bmatrix}$$

$$\underset{15\times 9}{[{}^{n}\mathbb{T}^{\gamma\alpha}_{2,4}]} := \begin{bmatrix}
T_{11|11|11|1111} & T_{11|22|11|1111} & \sqrt{2}T_{11|12|11|1111} & T_{22|11|11|1111} & T_{22|22|11|1111} & \sqrt{2}T_{22|12|11|1111} & \sqrt{2}T_{12|11|11|1111} & \sqrt{2}T_{12|22|11|1111} & \sqrt{4}T_{12|12|11|1111} \\
T_{11|11|22|1111} & T_{11|22|22|1111} & \sqrt{2}T_{11|12|22|1111} & T_{22|11|22|1111} & T_{22|22|22|1111} & \sqrt{2}T_{22|12|22|1111} & \sqrt{2}T_{12|11|22|1111} & \sqrt{2}T_{12|22|22|1111} & \sqrt{4}T_{12|12|22|1111} \\
\sqrt{2}T_{11|11|12|1111} & \sqrt{2}T_{11|22|12|1111} & \sqrt{4}T_{11|12|12|1111} & \sqrt{2}T_{22|11|12|1111} & \sqrt{2}T_{22|22|12|1111} & \sqrt{4}T_{22|12|12|1111} & \sqrt{4}T_{12|11|12|1111} & \sqrt{4}T_{12|22|12|1111} & \sqrt{8}T_{12|12|12|1111} \\
T_{11|11|11|2222} & T_{11|22|11|2222} & \sqrt{2}T_{11|12|11|2222} & T_{22|11|11|2222} & T_{22|22|11|2222} & \sqrt{2}T_{22|12|11|2222} & \sqrt{2}T_{12|11|11|2222} & \sqrt{2}T_{12|22|11|2222} & \sqrt{4}T_{12|12|11|2222} \\
T_{11|11|22|2222} & T_{11|22|22|2222} & \sqrt{2}T_{11|12|22|2222} & T_{22|11|22|2222} & T_{22|22|22|2222} & \sqrt{2}T_{22|12|22|2222} & \sqrt{2}T_{12|11|22|2222} & \sqrt{2}T_{12|22|22|2222} & \sqrt{4}T_{12|12|22|2222} \\
\sqrt{2}T_{11|11|12|2222} & \sqrt{2}T_{11|22|12|2222} & \sqrt{4}T_{11|12|12|2222} & \sqrt{2}T_{22|11|12|2222} & \sqrt{2}T_{22|22|12|2222} & \sqrt{4}T_{22|12|12|2222} & \sqrt{4}T_{12|11|12|2222} & \sqrt{4}T_{12|22|12|2222} & \sqrt{8}T_{12|12|12|2222} \\
\sqrt{4}T_{11|11|11|1112} & \sqrt{4}T_{11|22|11|1112} & \sqrt{8}T_{11|12|11|1112} & \sqrt{4}T_{22|11|11|1112} & \sqrt{4}T_{22|22|11|1112} & \sqrt{8}T_{22|12|11|1112} & \sqrt{8}T_{12|11|11|1112} & \sqrt{8}T_{12|22|11|1112} & \sqrt{24}T_{12|12|11|1112} \\
\sqrt{4}T_{11|11|22|1112} & \sqrt{4}T_{11|22|22|1112} & \sqrt{8}T_{11|12|22|1112} & \sqrt{4}T_{22|11|22|1112} & \sqrt{4}T_{22|22|22|1112} & \sqrt{8}T_{22|12|22|1112} & \sqrt{8}T_{12|11|22|1112} & \sqrt{8}T_{12|22|22|1112} & \sqrt{24}T_{12|12|22|1112} \\
\sqrt{8}T_{11|11|12|1112} & \sqrt{8}T_{11|22|12|1112} & \sqrt{16}T_{11|12|12|1112} & \sqrt{8}T_{22|11|12|1112} & \sqrt{8}T_{22|22|12|1112} & \sqrt{16}T_{22|12|12|1112} & \sqrt{16}T_{12|11|12|1112} & \sqrt{16}T_{12|22|12|1112} & \sqrt{32}T_{12|12|12|1112} \\
\sqrt{6}T_{11|11|11|1122} & \sqrt{6}T_{11|22|11|1122} & \sqrt{12}T_{11|12|11|1122} & \sqrt{6}T_{22|11|11|1122} & \sqrt{6}T_{22|22|11|1122} & \sqrt{12}T_{22|12|11|1122} & \sqrt{12}T_{12|11|11|1122} & \sqrt{12}T_{12|22|11|1122} & \sqrt{24}T_{12|12|11|1122} \\
\sqrt{6}T_{11|11|22|1122} & \sqrt{6}T_{11|22|22|1122} & \sqrt{12}T_{11|12|22|1122} & \sqrt{6}T_{22|11|22|1122} & \sqrt{6}T_{22|22|22|1122} & \sqrt{12}T_{22|12|22|1122} & \sqrt{12}T_{12|11|22|1122} & \sqrt{12}T_{12|22|22|1122} & \sqrt{24}T_{12|12|22|1122} \\
\sqrt{12}T_{11|11|12|1122} & \sqrt{12}T_{11|22|12|1122} & \sqrt{24}T_{11|12|12|1122} & \sqrt{12}T_{22|11|12|1122} & \sqrt{12}T_{22|22|12|1122} & \sqrt{24}T_{22|12|12|1122} & \sqrt{24}T_{12|11|12|1122} & \sqrt{24}T_{12|22|12|1122} & \sqrt{48}T_{12|12|12|1122} \\
\sqrt{4}T_{11|11|11|1222} & \sqrt{4}T_{11|22|11|1222} & \sqrt{8}T_{11|12|11|1222} & \sqrt{4}T_{22|11|11|1222} & \sqrt{4}T_{22|22|11|1222} & \sqrt{8}T_{22|12|11|1222} & \sqrt{8}T_{12|11|11|1222} & \sqrt{8}T_{12|22|11|1222} & \sqrt{24}T_{12|12|11|1222} \\
\sqrt{4}T_{11|11|22|1222} & \sqrt{4}T_{11|22|22|1222} & \sqrt{8}T_{11|12|22|1222} & \sqrt{4}T_{22|11|22|1222} & \sqrt{4}T_{22|22|22|1222} & \sqrt{8}T_{22|12|22|1222} & \sqrt{8}T_{12|11|22|1222} & \sqrt{8}T_{12|22|22|1222} & \sqrt{24}T_{12|12|22|1222} \\
\sqrt{8}T_{11|11|12|1222} & \sqrt{8}T_{11|22|12|1222} & \sqrt{16}T_{11|12|12|1222} & \sqrt{8}T_{22|11|12|1222} & \sqrt{8}T_{22|22|12|1222} & \sqrt{16}T_{22|12|12|1222} & \sqrt{16}T_{12|11|12|1222} & \sqrt{16}T_{12|22|12|1222} & \sqrt{32}T_{12|12|12|1222}
\end{bmatrix}$$

“Generalized Mandel representation” for assembly of a global system of stationarity equations

The components of the influence tensors ${}^n\mathbb{T}^{\gamma\alpha}_{s,4}$ are stored into matrices of the form / $r = 4$

$$T_{s_1 s_2 s_3|rs|ij|klmn} := ({}^nT^{\gamma\alpha}_{3,4})_{s_1 s_2 s_3 rsijklmn}$$

$$T_{s_1 s_2 s_3 s_4|rs|ij|klmn} := ({}^nT^{\gamma\alpha}_{4,4})_{s_1 s_2 s_3 s_4 rsijklmn}$$

$$\underset{15\times 12}{[{}^n\mathbb{T}^{\gamma\alpha}_{3,4}]} := \begin{bmatrix}
T_{111|11|11|1111} & T_{111|22|11|1111} & \sqrt{2}T_{111|12|11|1111} & T_{222|11|11|1111} & T_{222|22|11|1111} & \sqrt{2}T_{222|12|11|1111} & \sqrt{3}T_{112|11|11|1111} & \sqrt{3}T_{112|22|11|1111} & \sqrt{6}T_{112|12|11|1111} & \sqrt{3}T_{122|11|11|1111} & \sqrt{3}T_{122|22|11|1111} & \sqrt{6}T_{122|12|11|1111} \\
T_{111|11|22|1111} & T_{111|22|22|1111} & \sqrt{2}T_{111|12|22|1111} & T_{222|11|22|1111} & T_{222|22|22|1111} & \sqrt{2}T_{222|12|22|1111} & \sqrt{3}T_{112|11|22|1111} & \sqrt{3}T_{112|22|22|1111} & \sqrt{6}T_{112|12|22|1111} & \sqrt{3}T_{122|11|22|1111} & \sqrt{3}T_{122|22|22|1111} & \sqrt{6}T_{122|12|22|1111} \\
\sqrt{2}T_{111|11|12|1111} & \sqrt{2}T_{111|22|12|1111} & \sqrt{8}T_{111|12|12|1111} & \sqrt{2}T_{222|11|12|1111} & \sqrt{2}T_{222|22|12|1111} & \sqrt{8}T_{222|12|12|1111} & \sqrt{6}T_{112|11|12|1111} & \sqrt{6}T_{112|22|12|1111} & \sqrt{12}T_{112|12|12|1111} & \sqrt{6}T_{122|11|12|1111} & \sqrt{6}T_{122|22|12|1111} & \sqrt{12}T_{122|12|12|1111} \\
T_{111|11|11|2222} & T_{111|22|11|2222} & \sqrt{2}T_{111|12|11|2222} & T_{222|11|11|2222} & T_{222|22|11|2222} & \sqrt{2}T_{222|12|11|2222} & \sqrt{3}T_{112|11|11|2222} & \sqrt{3}T_{112|22|11|2222} & \sqrt{6}T_{112|12|11|2222} & \sqrt{3}T_{122|11|11|2222} & \sqrt{3}T_{122|22|11|2222} & \sqrt{6}T_{122|12|11|2222} \\
T_{111|11|22|2222} & T_{111|22|22|2222} & \sqrt{2}T_{111|12|22|2222} & T_{222|11|22|2222} & T_{222|22|22|2222} & \sqrt{2}T_{222|12|22|2222} & \sqrt{3}T_{112|11|22|2222} & \sqrt{3}T_{112|22|22|2222} & \sqrt{6}T_{112|12|22|2222} & \sqrt{3}T_{122|11|22|2222} & \sqrt{3}T_{122|22|22|2222} & \sqrt{6}T_{122|12|22|2222} \\
\sqrt{2}T_{111|11|12|2222} & \sqrt{2}T_{111|22|12|2222} & \sqrt{8}T_{111|12|12|2222} & \sqrt{2}T_{222|11|12|2222} & \sqrt{2}T_{222|22|12|2222} & \sqrt{8}T_{222|12|12|2222} & \sqrt{6}T_{112|11|12|2222} & \sqrt{6}T_{112|22|12|2222} & \sqrt{12}T_{112|12|12|2222} & \sqrt{6}T_{122|11|12|2222} & \sqrt{6}T_{122|22|12|2222} & \sqrt{12}T_{122|12|12|2222} \\
\sqrt{4}T_{111|11|11|1112} & \sqrt{4}T_{111|22|11|1112} & \sqrt{8}T_{111|12|11|1112} & \sqrt{4}T_{222|11|11|1112} & \sqrt{4}T_{222|22|11|1112} & \sqrt{8}T_{222|12|11|1112} & \sqrt{12}T_{112|11|11|1112} & \sqrt{12}T_{112|22|11|1112} & \sqrt{24}T_{112|12|11|1112} & \sqrt{12}T_{122|11|11|1112} & \sqrt{12}T_{122|22|11|1112} & \sqrt{24}T_{122|12|11|1112} \\
\sqrt{4}T_{111|11|22|1112} & \sqrt{4}T_{111|22|22|1112} & \sqrt{8}T_{111|12|22|1112} & \sqrt{4}T_{222|11|22|1112} & \sqrt{4}T_{222|22|22|1112} & \sqrt{8}T_{222|12|22|1112} & \sqrt{12}T_{112|11|22|1112} & \sqrt{12}T_{112|22|22|1112} & \sqrt{24}T_{112|12|22|1112} & \sqrt{12}T_{122|11|22|1112} & \sqrt{12}T_{122|22|22|1112} & \sqrt{24}T_{122|12|22|1112} \\
\sqrt{8}T_{111|11|12|1112} & \sqrt{8}T_{111|22|12|1112} & \sqrt{16}T_{111|12|12|1112} & \sqrt{8}T_{222|11|12|1112} & \sqrt{8}T_{222|22|12|1112} & \sqrt{16}T_{222|12|12|1112} & \sqrt{24}T_{112|11|12|1112} & \sqrt{24}T_{112|22|12|1112} & \sqrt{48}T_{112|12|12|1112} & \sqrt{24}T_{122|11|12|1112} & \sqrt{24}T_{122|22|12|1112} & \sqrt{48}T_{122|12|12|1112} \\
\sqrt{6}T_{111|11|11|1122} & \sqrt{6}T_{111|22|11|1122} & \sqrt{12}T_{111|12|11|1122} & \sqrt{6}T_{222|11|11|1122} & \sqrt{6}T_{222|22|11|1122} & \sqrt{12}T_{222|12|11|1122} & \sqrt{18}T_{112|11|11|1122} & \sqrt{18}T_{112|22|11|1122} & \sqrt{36}T_{112|12|11|1122} & \sqrt{18}T_{122|11|11|1122} & \sqrt{18}T_{122|22|11|1122} & \sqrt{36}T_{122|12|11|1122} \\
\sqrt{6}T_{111|11|22|1122} & \sqrt{6}T_{111|22|22|1122} & \sqrt{12}T_{111|12|22|1122} & \sqrt{6}T_{222|11|22|1122} & \sqrt{6}T_{222|22|22|1122} & \sqrt{12}T_{222|12|22|1122} & \sqrt{18}T_{112|11|22|1122} & \sqrt{18}T_{112|22|22|1122} & \sqrt{36}T_{112|12|22|1122} & \sqrt{18}T_{122|11|22|1122} & \sqrt{18}T_{122|22|22|1122} & \sqrt{36}T_{122|12|22|1122} \\
\sqrt{12}T_{111|11|12|1122} & \sqrt{12}T_{111|22|12|1122} & \sqrt{24}T_{111|12|12|1122} & \sqrt{12}T_{222|11|12|1122} & \sqrt{12}T_{222|22|12|1122} & \sqrt{24}T_{222|12|12|1122} & \sqrt{36}T_{112|11|12|1122} & \sqrt{36}T_{112|22|12|1122} & \sqrt{72}T_{112|12|12|1122} & \sqrt{36}T_{122|11|12|1122} & \sqrt{36}T_{122|22|12|1122} & \sqrt{72}T_{122|12|12|1122} \\
\sqrt{4}T_{111|11|11|1222} & \sqrt{4}T_{111|22|11|1222} & \sqrt{8}T_{111|12|11|1222} & \sqrt{4}T_{222|11|11|1222} & \sqrt{4}T_{222|22|11|1222} & \sqrt{8}T_{222|12|11|1222} & \sqrt{12}T_{112|11|11|1222} & \sqrt{12}T_{112|22|11|1222} & \sqrt{24}T_{112|12|11|1222} & \sqrt{12}T_{122|11|11|1222} & \sqrt{12}T_{122|22|11|1222} & \sqrt{24}T_{122|12|11|1222} \\
\sqrt{4}T_{111|11|22|1222} & \sqrt{4}T_{111|22|22|1222} & \sqrt{8}T_{111|12|22|1222} & \sqrt{4}T_{222|11|22|1222} & \sqrt{4}T_{222|22|22|1222} & \sqrt{8}T_{222|12|22|1222} & \sqrt{12}T_{112|11|22|1222} & \sqrt{12}T_{112|22|22|1222} & \sqrt{24}T_{112|12|22|1222} & \sqrt{12}T_{122|11|22|1222} & \sqrt{12}T_{122|22|22|1222} & \sqrt{24}T_{122|12|22|1222} \\
\sqrt{8}T_{111|11|12|1222} & \sqrt{8}T_{111|22|12|1222} & \sqrt{16}T_{111|12|12|1222} & \sqrt{8}T_{222|11|12|1222} & \sqrt{8}T_{222|22|12|1222} & \sqrt{16}T_{222|12|12|1222} & \sqrt{24}T_{112|11|12|1222} & \sqrt{24}T_{112|22|12|1222} & \sqrt{48}T_{112|12|12|1222} & \sqrt{24}T_{122|11|12|1222} & \sqrt{24}T_{122|22|12|1222} & \sqrt{48}T_{122|12|12|1222}
\end{bmatrix}$$

$$\underset{15\times 15}{[{}^n\mathbb{T}^{\gamma\alpha}_{4,4}]} := \begin{bmatrix}
T_{1111|11|11|1111} & T_{1111|22|11|1111} & \sqrt{2}T_{1111|12|11|1111} & T_{2222|11|11|1111} & T_{2222|22|11|1111} & \sqrt{2}T_{2222|12|11|1111} & \sqrt{4}T_{1112|11|11|1111} & \sqrt{4}T_{1112|22|11|1111} & \sqrt{8}T_{1112|12|11|1111} & \sqrt{6}T_{1122|11|11|1111} & \sqrt{6}T_{1122|22|11|1111} & \sqrt{12}T_{1122|12|11|1111} & \sqrt{4}T_{1222|11|11|1111} & \sqrt{4}T_{1222|22|11|1111} & \sqrt{8}T_{1222|12|11|1111} \\
T_{1111|11|22|1111} & T_{1111|22|22|1111} & \sqrt{2}T_{1111|12|22|1111} & T_{2222|11|22|1111} & T_{2222|22|22|1111} & \sqrt{2}T_{2222|12|22|1111} & \sqrt{4}T_{1112|11|22|1111} & \sqrt{4}T_{1112|22|22|1111} & \sqrt{8}T_{1112|12|22|1111} & \sqrt{6}T_{1122|11|22|1111} & \sqrt{6}T_{1122|22|22|1111} & \sqrt{12}T_{1122|12|22|1111} & \sqrt{4}T_{1222|11|22|1111} & \sqrt{4}T_{1222|22|22|1111} & \sqrt{8}T_{1222|12|22|1111} \\
\sqrt{2}T_{1111|11|12|1111} & \sqrt{2}T_{1111|22|12|1111} & \sqrt{4}T_{1111|12|12|1111} & \sqrt{2}T_{2222|11|12|1111} & \sqrt{2}T_{2222|22|12|1111} & \sqrt{4}T_{2222|12|12|1111} & \sqrt{8}T_{1112|11|12|1111} & \sqrt{8}T_{1112|22|12|1111} & \sqrt{16}T_{1112|12|12|1111} & \sqrt{12}T_{1122|11|12|1111} & \sqrt{12}T_{1122|22|12|1111} & \sqrt{24}T_{1122|12|12|1111} & \sqrt{8}T_{1222|11|12|1111} & \sqrt{8}T_{1222|22|12|1111} & \sqrt{16}T_{1222|12|12|1111} \\
T_{1111|11|11|2222} & T_{1111|22|11|2222} & \sqrt{2}T_{1111|12|11|2222} & T_{2222|11|11|2222} & T_{2222|22|11|2222} & \sqrt{2}T_{2222|12|11|2222} & \sqrt{4}T_{1112|11|11|2222} & \sqrt{4}T_{1112|22|11|2222} & \sqrt{8}T_{1112|12|11|2222} & \sqrt{6}T_{1122|11|11|2222} & \sqrt{6}T_{1122|22|11|2222} & \sqrt{12}T_{1122|12|11|2222} & \sqrt{4}T_{1222|11|11|2222} & \sqrt{4}T_{1222|22|11|2222} & \sqrt{8}T_{1222|12|11|2222} \\
T_{1111|11|22|2222} & T_{1111|22|22|2222} & \sqrt{2}T_{1111|12|22|2222} & T_{2222|11|22|2222} & T_{2222|22|22|2222} & \sqrt{2}T_{2222|12|22|2222} & \sqrt{4}T_{1112|11|22|2222} & \sqrt{4}T_{1112|22|22|2222} & \sqrt{8}T_{1112|12|22|2222} & \sqrt{6}T_{1122|11|22|2222} & \sqrt{6}T_{1122|22|22|2222} & \sqrt{12}T_{1122|12|22|2222} & \sqrt{4}T_{1222|11|22|2222} & \sqrt{4}T_{1222|22|22|2222} & \sqrt{8}T_{1222|12|22|2222} \\
\sqrt{2}T_{1111|11|12|2222} & \sqrt{2}T_{1111|22|12|2222} & \sqrt{4}T_{1111|12|12|2222} & \sqrt{2}T_{2222|11|12|2222} & \sqrt{2}T_{2222|22|12|2222} & \sqrt{4}T_{2222|12|12|2222} & \sqrt{8}T_{1112|11|12|2222} & \sqrt{8}T_{1112|22|12|2222} & \sqrt{16}T_{1112|12|12|2222} & \sqrt{12}T_{1122|11|12|2222} & \sqrt{12}T_{1122|22|12|2222} & \sqrt{24}T_{1122|12|12|2222} & \sqrt{8}T_{1222|11|12|2222} & \sqrt{8}T_{1222|22|12|2222} & \sqrt{16}T_{1222|12|12|2222} \\
\sqrt{4}T_{1111|11|11|1112} & \sqrt{4}T_{1111|22|11|1112} & \sqrt{8}T_{1111|12|11|1112} & \sqrt{4}T_{2222|11|11|1112} & \sqrt{4}T_{2222|22|11|1112} & \sqrt{8}T_{2222|12|11|1112} & \sqrt{16}T_{1112|11|11|1112} & \sqrt{16}T_{1112|22|11|1112} & \sqrt{36}T_{1112|12|11|1112} & \sqrt{24}T_{1122|11|11|1112} & \sqrt{24}T_{1122|22|11|1112} & \sqrt{48}T_{1122|12|11|1112} & \sqrt{16}T_{1222|11|11|1112} & \sqrt{16}T_{1222|22|11|1112} & \sqrt{32}T_{1222|12|11|1112} \\
\sqrt{4}T_{1111|11|22|1112} & \sqrt{4}T_{1111|22|22|1112} & \sqrt{8}T_{1111|12|22|1112} & \sqrt{4}T_{2222|11|22|1112} & \sqrt{4}T_{2222|22|22|1112} & \sqrt{8}T_{2222|12|22|1112} & \sqrt{16}T_{1112|11|22|1112} & \sqrt{16}T_{1112|22|22|1112} & \sqrt{36}T_{1112|12|22|1112} & \sqrt{24}T_{1122|11|22|1112} & \sqrt{24}T_{1122|22|22|1112} & \sqrt{48}T_{1122|12|22|1112} & \sqrt{16}T_{1222|11|22|1112} & \sqrt{16}T_{1222|22|22|1112} & \sqrt{32}T_{1222|12|22|1112} \\
\sqrt{8}T_{1111|11|12|1112} & \sqrt{8}T_{1111|22|12|1112} & \sqrt{16}T_{1111|12|12|1112} & \sqrt{8}T_{2222|11|12|1112} & \sqrt{8}T_{2222|22|12|1112} & \sqrt{16}T_{2222|12|12|1112} & \sqrt{32}T_{1112|11|12|1112} & \sqrt{32}T_{1112|22|12|1112} & \sqrt{64}T_{1112|12|12|1112} & \sqrt{48}T_{1122|11|12|1112} & \sqrt{48}T_{1122|22|12|1112} & \sqrt{96}T_{1122|12|12|1112} & \sqrt{32}T_{1222|11|12|1112} & \sqrt{32}T_{1222|22|12|1112} & \sqrt{64}T_{1222|12|12|1112} \\
\sqrt{6}T_{1111|11|11|1122} & \sqrt{6}T_{1111|22|11|1122} & \sqrt{12}T_{1111|12|11|1122} & \sqrt{6}T_{2222|11|11|1122} & \sqrt{6}T_{2222|22|11|1122} & \sqrt{12}T_{2222|12|11|1122} & \sqrt{24}T_{1112|11|11|1122} & \sqrt{24}T_{1112|22|11|1122} & \sqrt{48}T_{1112|12|11|1122} & \sqrt{36}T_{1122|11|11|1122} & \sqrt{36}T_{1122|22|11|1122} & \sqrt{72}T_{1122|12|11|1122} & \sqrt{24}T_{1222|11|11|1122} & \sqrt{24}T_{1222|22|11|1122} & \sqrt{48}T_{1222|12|11|1122} \\
\sqrt{6}T_{1111|11|22|1122} & \sqrt{6}T_{1111|22|22|1122} & \sqrt{12}T_{1111|12|22|1122} & \sqrt{6}T_{2222|11|22|1122} & \sqrt{6}T_{2222|22|22|1122} & \sqrt{12}T_{2222|12|22|1122} & \sqrt{24}T_{1112|11|22|1122} & \sqrt{24}T_{1112|22|22|1122} & \sqrt{48}T_{1112|12|22|1122} & \sqrt{36}T_{1122|11|22|1122} & \sqrt{36}T_{1122|22|22|1122} & \sqrt{72}T_{1122|12|22|1122} & \sqrt{24}T_{1222|11|22|1122} & \sqrt{24}T_{1222|22|22|1122} & \sqrt{48}T_{1222|12|22|1122} \\
\sqrt{12}T_{1111|11|12|1122} & \sqrt{12}T_{1111|22|12|1122} & \sqrt{24}T_{1111|12|12|1122} & \sqrt{12}T_{2222|11|12|1122} & \sqrt{12}T_{2222|22|12|1122} & \sqrt{24}T_{2222|12|12|1122} & \sqrt{48}T_{1112|11|12|1122} & \sqrt{48}T_{1112|22|12|1122} & \sqrt{96}T_{1112|12|12|1122} & \sqrt{72}T_{1122|11|12|1122} & \sqrt{72}T_{1122|22|12|1122} & \sqrt{144}T_{1122|12|12|1122} & \sqrt{48}T_{1222|11|12|1122} & \sqrt{48}T_{1222|22|12|1122} & \sqrt{96}T_{1222|12|12|1122} \\
\sqrt{4}T_{1111|11|11|1222} & \sqrt{4}T_{1111|22|11|1222} & \sqrt{8}T_{1111|12|11|1222} & \sqrt{4}T_{2222|11|11|1222} & \sqrt{4}T_{2222|22|11|1222} & \sqrt{8}T_{2222|12|11|1222} & \sqrt{16}T_{1112|11|11|1222} & \sqrt{16}T_{1112|22|11|1222} & \sqrt{36}T_{1112|12|11|1222} & \sqrt{24}T_{1122|11|11|1222} & \sqrt{24}T_{1122|22|11|1222} & \sqrt{48}T_{1122|12|11|1222} & \sqrt{16}T_{1222|11|11|1222} & \sqrt{16}T_{1222|22|11|1222} & \sqrt{32}T_{1222|12|11|1222} \\
\sqrt{4}T_{1111|11|22|1222} & \sqrt{4}T_{1111|22|22|1222} & \sqrt{8}T_{1111|12|22|1222} & \sqrt{4}T_{2222|11|22|1222} & \sqrt{4}T_{2222|22|22|1222} & \sqrt{8}T_{2222|12|22|1222} & \sqrt{16}T_{1112|11|22|1222} & \sqrt{16}T_{1112|22|22|1222} & \sqrt{36}T_{1112|12|22|1222} & \sqrt{24}T_{1122|11|22|1222} & \sqrt{24}T_{1122|22|22|1222} & \sqrt{48}T_{1122|12|22|1222} & \sqrt{16}T_{1222|11|22|1222} & \sqrt{16}T_{1222|22|22|1222} & \sqrt{32}T_{1222|12|22|1222} \\
\sqrt{8}T_{1111|11|12|1222} & \sqrt{8}T_{1111|22|12|1222} & \sqrt{16}T_{1111|12|12|1222} & \sqrt{8}T_{2222|11|12|1222} & \sqrt{8}T_{2222|22|12|1222} & \sqrt{16}T_{2222|12|12|1222} & \sqrt{32}T_{1112|11|12|1222} & \sqrt{32}T_{1112|22|12|1222} & \sqrt{64}T_{1112|12|12|1222} & \sqrt{48}T_{1122|11|12|1222} & \sqrt{48}T_{1122|22|12|1222} & \sqrt{96}T_{1122|12|12|1222} & \sqrt{32}T_{1222|11|12|1222} & \sqrt{32}T_{1222|22|12|1222} & \sqrt{64}T_{1222|12|12|1222}
\end{bmatrix}$$

“Generalized Mandel representation” for assembly of a global system of stationarity equations

- *Global influence matrices* are assembled as follows,

```
for I in [0… 3(r+1)nα -1]:
   for J in [0… 3(s+1)nα -1]:
    ddim_i=3(r+1), ki=I%ddim_i, α =(I-ki)/ddim_i
    ddim_j=3(s+1), kj=J%ddim_j, γ =(J-kj)/ddim_j
     T_global(s,r,I=3(2)(r+1)… 3(2+1)(r+1)-1,
                 J=3(1)(s+1)… 3(1+1)(s+1)-1)
     ⇓
     T_local(γ ,α ,s=2,r=1,ki,kj)
     ⇓
     T_sym_infl(γ ,α ,ns1,ns2,ijkl,nr1,nr2)
     =
     T_sym_infl(α ,γ ,nr1,nr2,ijkl,ns1,ns2)
```

Generally not symmetric

True if $\alpha \neq \gamma$

$$[\mathbb{T}_{s,r}] := \begin{bmatrix} [^n\mathbb{T}^{0,0}_{s,r}] & [^n\mathbb{T}^{1,0}_{s,r}] & [^n\mathbb{T}^{2,0}_{s,r}] & \cdots & [^n\mathbb{T}^{n_\alpha-1,0}_{s,r}] \\ [^n\mathbb{T}^{0,1}_{s,r}] & [^n\mathbb{T}^{1,1}_{s,r}] & [^n\mathbb{T}^{2,1}_{s,r}] & \cdots & [^n\mathbb{T}^{n_\alpha-1,1}_{s,r}] \\ [^n\mathbb{T}^{0,2}_{s,r}] & [^n\mathbb{T}^{1,2}_{s,r}] & [^n\mathbb{T}^{2,2}_{s,r}] & \cdots & [^n\mathbb{T}^{n_\alpha-1,2}_{s,r}] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [^n\mathbb{T}^{0,n_\alpha-1}_{s,r}] & \cdots & \cdots & \cdots & [^n\mathbb{T}^{n_\alpha-1,n_\alpha-1}_{s,r}] \end{bmatrix}$$

$3(r+1)n_\alpha \times 3(s+1)n_\alpha$

$3(r+1) \times 3(s+1)$

Remarks on symmetry: $({}^nT^{\alpha\alpha}_{r,s})_{r_1\ldots r_r ijkl s_1\ldots s_s} = ({}^nT^{\alpha\alpha}_{r,s})_{r_1\ldots r_r klij s_1\ldots s_s} \implies [^n\mathbb{T}^{\alpha\alpha}_{s,r}]^T = [^n\mathbb{T}^{\alpha\alpha}_{s,r}]^T$

$$\Omega'_\gamma = \Omega'_\alpha \implies {}^n\mathbb{T}^{\gamma\alpha}_{r,s} = {}^n\mathbb{T}^{\gamma\alpha}_{r,s}$$

$$\Omega'_\gamma = \Omega'_\alpha \;\forall\, \alpha,\gamma \in [0, n_\alpha - 1] \implies [^n\mathbb{T}_{r,s}] = [^n\mathbb{T}_{r,s}]^T$$

- Recall the *global Minkowski weighted compliance matrices*

$$[\mathbb{M}_{0,0}] := \begin{bmatrix} c_0[\Delta\mathbb{M}^0] & 0 & \cdots & 0 \\ 0 & c_1[\Delta\mathbb{M}^1] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n_\alpha-1}[\Delta\mathbb{M}^{n_\alpha-1}] \end{bmatrix}$$

$3n_\alpha \times 3n_\alpha$; 3×3

$$[\mathbb{M}_{s,r}] := \begin{bmatrix} [\Delta\mathbb{M}^0 \otimes \mathcal{W}^{s+r,0}_0] & 0 & \cdots & 0 \\ 0 & [\Delta\mathbb{M}^1 \otimes \mathcal{W}^{s+r,0}_0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [\Delta\mathbb{M}^{n_\alpha-1} \otimes \mathcal{W}^{s+r,0}_0] \end{bmatrix}$$

$3n_\alpha(r+1)\times 3n_\alpha(s+1)$; $3(r+1) \times 3(s+1)$

```
ddim_i=3(r+1), ki=I%ddim_i, α =(I-ki)/ddim_i
ddim_j=3(s+1), kj=J%ddim_j
dMW_local(α,s,r,ki,kj) ⇐ M_global(s,r,I=3(1)(r+1)… 3(1+1)(r+1)-1),
                                   J=3(1)(s+1)… 3(1+1)(s+1)-1))
```

Remarks on symmetry: $[\Delta\mathbb{M}^\alpha \otimes \mathcal{W}^{r+s,0}_0] = [\Delta\mathbb{M}^\alpha \otimes \mathcal{W}^{s+r,0}_0]^T \implies [\mathbb{M}_{r,s}] = [\mathbb{M}_{s,r}]^T$

- We define $\mathbb{D}^r_s := [\mathbb{M}_{s,r}] + [\mathbb{T}_{s,r}]$ and pose the “*r stationarity equations*” in matrix form,

$$\{\bar{\varepsilon}^r\} = [\mathbb{D}^r_1]\{\boldsymbol{\partial\tau}\} + [\mathbb{D}^r_2]\{\boldsymbol{\partial}^2\boldsymbol{\tau}\} + [\mathbb{D}^r_3]\{\boldsymbol{\partial}^3\boldsymbol{\tau}\} + \cdots + [\mathbb{D}^r_p]\{\boldsymbol{\partial}^p\boldsymbol{\tau}\}$$

$3(r+1)n_\alpha \times 1$, $3(r+1)n_\alpha \times 6n_\alpha$, $3(r+1)n_\alpha \times 9n_\alpha$, $3(r+1)n_\alpha \times 12n_\alpha$, $3(r+1)n_\alpha \times 3(p+1)n_\alpha$

"Generalized Mandel representation" for assembly of a global system of stationarity equations

We want to solve the system

$$r=0 \quad \rightarrow \underset{3n_\alpha\times 1}{\{\overline{\varepsilon}^0\}} = \underset{3n_\alpha\times 3n_\alpha}{[\mathbb{D}^0_0]}\underset{3n_\alpha\times 1}{\{\boldsymbol{\tau}\}}$$

$$r=1 \quad \rightarrow \underset{6n_\alpha\times 1}{\{\overline{\varepsilon}^1\}} = \underset{6n_\alpha\times 6n_\alpha}{[\mathbb{D}^1_1]}\underset{6n_\alpha\times 1}{\{\boldsymbol{\partial\tau}\}} + \underset{6n_\alpha\times 9n_\alpha}{[\mathbb{D}^1_2]}\underset{9n_\alpha\times 1}{\{\boldsymbol{\partial}^2\boldsymbol{\tau}\}} + \underset{6n_\alpha\times 12n_\alpha}{[\mathbb{D}^1_3]}\underset{12n_\alpha\times 1}{\{\boldsymbol{\partial}^3\boldsymbol{\tau}\}} + \cdots + \underset{6n_\alpha\times 3(p+1)n_\alpha}{[\mathbb{D}^1_p]}\overset{3(p+1)n_\alpha\times 1}{\{\boldsymbol{\partial}^p\boldsymbol{\tau}\}}$$

$$r=2 \quad \rightarrow \underset{9n_\alpha\times 1}{\{\overline{\varepsilon}^2\}} = \underset{9n_\alpha\times 6n_\alpha}{[\mathbb{D}^2_1]}\underset{6n_\alpha\times 1}{\{\boldsymbol{\partial\tau}\}} + \underset{9n_\alpha\times 9n_\alpha}{[\mathbb{D}^2_2]}\underset{9n_\alpha\times 1}{\{\boldsymbol{\partial}^2\boldsymbol{\tau}\}} + \underset{9n_\alpha\times 12n_\alpha}{[\mathbb{D}^2_3]}\underset{12n_\alpha\times 1}{\{\boldsymbol{\partial}^3\boldsymbol{\tau}\}} + \cdots + \underset{9n_\alpha\times 3(p+1)n_\alpha}{[\mathbb{D}^2_p]}\overset{3(p+1)n_\alpha\times 1}{\{\boldsymbol{\partial}^p\boldsymbol{\tau}\}}$$

$$r=3 \quad \rightarrow \underset{12n_\alpha\times 1}{\{\overline{\varepsilon}^3\}} = \underset{12n_\alpha\times 6n_\alpha}{[\mathbb{D}^3_1]}\underset{6n_\alpha\times 1}{\{\boldsymbol{\partial\tau}\}} + \underset{12n_\alpha\times 9n_\alpha}{[\mathbb{D}^3_2]}\underset{9n_\alpha\times 1}{\{\boldsymbol{\partial}^2\boldsymbol{\tau}\}} + \underset{12n_\alpha\times 12n_\alpha}{[\mathbb{D}^3_3]}\underset{12n_\alpha\times 1}{\{\boldsymbol{\partial}^3\boldsymbol{\tau}\}} + \cdots + \underset{12n_\alpha\times 3(p+1)n_\alpha}{[\mathbb{D}^3_p]}\overset{3(p+1)n_\alpha\times 1}{\{\boldsymbol{\partial}^p\boldsymbol{\tau}\}}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \vdots$$

$$r=p \quad \rightarrow \underset{3(p+1)n_\alpha\times 1}{\{\overline{\varepsilon}^p\}} = \underset{3(p+1)n_\alpha\times 6n_\alpha}{[\mathbb{D}^p_1]}\overset{6n_\alpha\times 1}{\{\boldsymbol{\partial\tau}\}} + \underset{3(p+1)n_\alpha\times 9n_\alpha}{[\mathbb{D}^p_2]}\overset{9n_\alpha\times 1}{\{\boldsymbol{\partial}^2\boldsymbol{\tau}\}} + \underset{3(p+1)n_\alpha\times 12n_\alpha}{[\mathbb{D}^p_3]}\overset{12n_\alpha\times 1}{\{\boldsymbol{\partial}^3\boldsymbol{\tau}\}} + \cdots + \underset{3(p+1)n_\alpha\times 3(p+1)n_\alpha}{[\mathbb{D}^p_p]}\overset{3(p+1)n_\alpha\times 1}{\{\boldsymbol{\partial}^p\boldsymbol{\tau}\}}$$

which we recast in

$$\underset{\frac{3n_\alpha}{2}(p^2+3p)\times 1}{\begin{Bmatrix}\{\overline{\varepsilon}^1\}\\ \{\overline{\varepsilon}^2\}\\ \{\overline{\varepsilon}^2\}\\ \vdots\\ \{\overline{\varepsilon}^p\}\end{Bmatrix}} = \underset{\frac{3n_\alpha}{2}(p^2+3p)\times\frac{3n_\alpha}{2}(p^2+3p)}{\begin{bmatrix}[\mathbb{D}^1_1] & [\mathbb{D}^1_2] & [\mathbb{D}^1_3] & \cdots & [\mathbb{D}^1_p]\\ [\mathbb{D}^2_1] & [\mathbb{D}^2_2] & [\mathbb{D}^2_3] & \cdots & [\mathbb{D}^2_p]\\ [\mathbb{D}^3_1] & [\mathbb{D}^3_2] & [\mathbb{D}^3_3] & \cdots & [\mathbb{D}^3_p]\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ [\mathbb{D}^p_1] & [\mathbb{D}^p_2] & [\mathbb{D}^p_3] & \cdots & [\mathbb{D}^p_p]\end{bmatrix}} \underset{\frac{3n_\alpha}{2}(p^2+3p)\times 1}{\begin{Bmatrix}\{\boldsymbol{\partial\tau}\}\\ \{\boldsymbol{\partial}^2\boldsymbol{\tau}\}\\ \{\boldsymbol{\partial}^3\boldsymbol{\tau}\}\\ \vdots\\ \{\boldsymbol{\partial}^p\boldsymbol{\tau}\}\end{Bmatrix}}$$

```
D_mat_assemble()

for r in [1… p]:
   for s in [1… p]:
      for i in [0… 3(r+1)nα -1]:
         for j in [0… 3(s+1)nα -1]:
          D[3nα((r-1)²+3(r-1))/2+i][3nα((s-1)²+3(s-1))/2+j] =
                 T_global(s,r,i,j) + M_global(s,r,i,j)
```

```
M_global(s=3,r=2,I=0… 3(2+1)nα -1,
                   J=0… 3(3+1)nα -1)
+
T_global(s=3,r=2,I=0… 3(2+1)nα -1,
                   J=0… 3(3+1)nα -1)
```

2D Stroh Formalism

- After Eshelby et al. (1953), Stroh (1958,1962) established the following framework to solve for displacement fields in 2D elastic anisotropic media. Assuming a superposition of solutions of the form

$u_i(\underline{x}) = a_i f(z)$ where $z = x_1 + p x_2$ with p complex,

we have $u_{k,sj}(\underline{x}) = \partial_j[(\delta_{s1} + p\delta_{s2}) a_k f'(z)] = (\delta_{s1} + p\delta_{s2}) a_k \partial_j[f'(z)] = (\delta_{s1} + p\delta_{s2}) a_k f''(z) \partial_j[z]$

$$= (\delta_{j1} + p\delta_{j2})(\delta_{s1} + p\delta_{s2}) a_k f''(z)$$

$$\begin{aligned} \partial_j[f^{(n)}(z)] &= f^{(n+1)}(z)\partial_j[z] \\ &= (\delta_{j1} + p\delta_{j2}) f^{(n+1)}(z) \end{aligned}$$

$$u_{k,s}(\underline{x}) = \partial_s[a_k f(z)] = a_k f'(z) \partial_s[z] = (\delta_{s1} + p\delta_{s2}) a_k f'(z)$$

so that the local statement of equilibrium becomes

$$\sigma_{ij}(\underline{x}) = L^0_{ijks} u_{k,s}(\underline{x})$$

$$L^0_{ijks} u_{k,sj}(\underline{x}) = 0 \;\forall\, i, \underline{x}$$

$$L^0_{ijks}(\delta_{j1} + p\delta_{j2})(\delta_{s1} + p\delta_{s2}) a_k f''(z) = 0$$

$$L^0_{ijks}(\delta_{j1} + p\delta_{j2})(\delta_{s1} + p\delta_{s2}) a_k = 0$$

$$[L^0_{i1k1} + p(L^0_{i1k2} + L^0_{i2k1}) + p^2 L^0_{i2k2}] a_k = 0$$

- Non-trivial solutions then satisfy

$$P(p) := \sum_{k=0}^{4} P_k p^k = 0$$

$$\begin{aligned} P_0 &= L^0_{1111} L^0_{1212} - L^0_{1112} L^0_{1112} \\ P_1 &= 2(L^0_{1111} L^0_{2212} - L^0_{1112} L^0_{1122}) \\ P_2 &= L^0_{1111} L^0_{2222} + 2(L^0_{1112} L^0_{2212} - L^0_{1122} L^0_{1212}) - L^0_{1122} L^0_{1122} \\ P_3 &= 2(L^0_{1112} L^0_{2222} - L^0_{1122} L^0_{2212}) \\ P_4 &= L^0_{1212} L^0_{2222} - L^0_{2212} L^0_{2212} \end{aligned}$$

2D Stroh eigensystem

$$\{(p_\alpha, \overline{p}_\alpha, \underline{a}^\alpha, \underline{\overline{a}}^\alpha) \mid P(p_\alpha) = 0,\ \Im\{p_\alpha\} > 0,\ \alpha = 1,2\}$$

$$[L^0_{i1k1} + p_\alpha(L^0_{i1k2} + L^0_{i2k1}) + p_\alpha^2 L^0_{i2k2}] a^\alpha_k = 0$$

(Not a regular eigenvalue problem)

2D Stroh Formalism

- For non-degenerate material symmetries, i.e. with independent Stroh eigenvectors, complete solutions for the displacement take the form

$$\underline{u}(\underline{x}) = \underline{a}^1 f_1(z_1) + \overline{\underline{a}}^1 f_3(\overline{z}_1) + \underline{a}^2 f_2(z_2) + \overline{\underline{a}}^2 f_4(\overline{z}_2)$$

where f_α are arbitrary functions (depending on BCs) and $z_\alpha := x_1 + p_\alpha x_2$.

- By linear elasticity, we have $\sigma_{i1} = (Q^0_{ik} + pR^0_{ik})a_k f'(z)$, $\sigma_{i2} = (R^0_{ki} + pT^0_{ik})a_k f'(z)$.
- Since local equilibrium requires $Q^0_{ik} + p(R^0_{ik} + R^0_{ki}) + p^2 T^0_{ik} = 0 \;\forall i$ $\implies R^0_{ki} + pT^0_{ik} = -\frac{1}{p}(Q^0_{ik} + pR^0_{ik})$,

$$Q^0_{ik} := L^0_{i1k1} \qquad R^0_{ik} := L^0_{i1k2} \qquad T^0_{ik} := L^0_{i2k2}$$

we have $\sigma_{i1} = (Q^0_{ik} + pR^0_{ik})a_k f' = -p(R^0_{ki} + pT^0_{ik})a_k f'(z)$ and $\sigma_{i2} = (R^0_{ki} + pT^0_{ik})a_k f'(z) = -(1/p)(Q^0_{ik} + pR^0_{ik})a_k f'(z)$

which we recast in $\sigma_{i1} = -pb_i f'(z)$, $\sigma_{i2} = b_i f'(z)$ → $b_i = (R^0_{ki} + pT^0_{ik})a_k = -\frac{1}{p}(Q^0_{ik} + pR^0_{ik})a_k$

- Then, stress functions $\varphi_i(z) = b_i f(z)$ are such that

$$\varphi_{i,j}(z) = b_i(\delta_{j1} + p\delta_{j2})f'(z) \implies \varphi_{i,1}(z) = b_i f'(z) = \sigma_{i2}(z), \quad \varphi_{i,2}(z) = pb_i f'(z) = -\sigma_{i1}(z)$$

and

$$\sigma_{12} = \sigma_{21} \implies \varphi_{1,1} + \varphi_{2,2} = 0, \quad (b_1 + pb_2)f'(z) = 0, \quad b_1 + pb_2 = 0$$

- Still under the assumption of non-degenerate symmetry, we have $\underline{\varphi}(\underline{x}) = \underline{b}^1 f_1(z_1) + \overline{\underline{b}}^1 f_3(\overline{z}_1) + \underline{b}^2 f_2(z_2) + \overline{\underline{b}}^2 f_4(\overline{z}_2)$.
- Solutions of the form $f_1(z_1) = q_1 f(z_1)$, $f_2(z_2) = q_2 f(z_2)$, $f_3(\overline{z}_1) = \overline{q}_1 \overline{f}(\overline{z}_1)$, $f_4(\overline{z}_2) = \overline{q}_2 \overline{f}(\overline{z}_2)$ are used.
- Since $2\Re\{\underline{a}^\alpha q_\alpha f(z_\alpha)\} = \underline{a}^\alpha q_\alpha f(z_\alpha) + \overline{\underline{a}}^\alpha \overline{q}_\alpha f(\overline{z}_\alpha)$, $2\Re\{\underline{b}^\alpha q_\alpha f(z_\alpha)\} = \underline{b}^\alpha q_\alpha f(z_\alpha) + \overline{\underline{b}}^\alpha \overline{q}_\alpha f(\overline{z}_\alpha)$ we have →

$$\underline{u}(\underline{x}) = 2\Re\{\underline{a}^1 f(z_1)q_1 + \underline{a}^2 f(z_2)q_2\}, \quad \underline{\varphi}(\underline{x}) = 2\Re\{\underline{b}^1 f(z_1)q_1 + \underline{b}^2 f(z_2)q_2\}$$

- If q_α is replaced by $-\mathrm{i}q_\alpha$, $\Re\{-\mathrm{i}z\} = \Im\{z\} \implies$

$$\underline{u}(\underline{x}) = 2\Im\{\underline{a}^1 f(z_1)q_1 + \underline{a}^2 f(z_2)q_2\}, \quad \underline{\varphi}(\underline{x}) = 2\Im\{\underline{b}^1 f(z_1)q_1 + \underline{b}^2 f(z_2)q_2\}$$

2D Stroh Formalism

- The function $f : z \mapsto \mathbb{C}$ and the complex coefficients q_α for $\alpha = 1, 2$ are solved for specific boundary conditions.
- To solve for Green functions,

$$\oint_{\mathcal{C}} \sigma_{ij}(\underline{x}) n_j(\underline{x}) \mathrm{d}s = \oint_{\mathcal{C}} \frac{\mathrm{d}\varphi_i(\underline{x})}{\mathrm{d}s} \mathrm{d}s = \varphi_i(s_b) - \varphi_i(s_a) = f_i \ \forall \mathcal{C} \subset \mathbb{R}^2 \text{ s.t. } \underline{0} \in \overline{\mathcal{C}}$$

- All free bodies containing the material point of application of the concentrated force $\underline{f}$ are in equilibrium,

$$\lim_{\|\underline{x}\| \to \infty} \sigma_{ij} = 0$$

- The medium is an infinitely large traction-free plane.

A concentrated force $\underline{f}$ is applied at $\underline{x} = \underline{0}$.

- Let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ with $r > 0$, $-\pi < \theta < \pi$ so that

$$\ln(z) = \begin{cases} \ln(r) & \text{if } \theta = 0, \\ \ln(r) \pm \mathrm{i}\pi & \text{if } \theta = \pm\pi \end{cases} \implies \ln(z)|_{\theta=\pi} - \ln(z)|_{\theta=-\pi} = 2\pi\mathrm{i}$$

- Redefine q_α s.t. $\underline{u}(\underline{x}) = \frac{1}{\pi}\Im\{\underline{a}^1 f(z_1) q_1^\infty + \underline{a}^2 f(z_2) q_2^\infty\}$ and $\underline{\varphi}(\underline{x}) = \frac{1}{\pi}\Im\{\underline{b}^1 f(z_1) q_1^\infty + \underline{b}^2 f(z_2) q_2^\infty\}$

then $f(z_\alpha) = \ln(z_\alpha) \implies$

$$\sum_{\alpha=1}^{2} \underline{b}^\alpha q_\alpha^\infty [f(z_\alpha)|_{\theta=\pi} - f(z_\alpha)|_{\theta=-\pi}] = 2\pi\mathrm{i}(\underline{b}^1 q_1^\infty + \underline{b}^2 q_2^\infty)$$

$$\implies \Im\left(\sum_{\alpha=1}^{2} \underline{b}^\alpha q_\alpha^\infty [f(z_\alpha)|_{\theta=\pi} - f(z_\alpha)|_{\theta=-\pi}]\right) = 2\pi\Re\{\underline{b}^1 q_1^\infty + \underline{b}^2 q_2^\infty\}$$

and $\underline{\varphi}(r,\pi) - \underline{\varphi}(r,-\pi) = \underline{f} \implies 2\Re\{\underline{b}^1 q_1^\infty + \underline{b}^2 q_2^\infty\} = \underline{f} \implies \sum_{\alpha=1}^{2} \left(\underline{b}^\alpha q_\alpha^\infty + \overline{\underline{b}}^\alpha \overline{q}_\alpha^\infty\right) = \underline{f}$.

- Similarly, by compatibility, we have:

$$\underline{u}(r,\pi) - \underline{u}(r,-\pi) = \underline{0} \implies 2\Re\{\underline{a}^1 q_1^\infty + \underline{a}^2 q_2^\infty\} = \underline{0} \implies \sum_{\alpha=1}^{2} (\underline{a}^\alpha q_\alpha^\infty + \overline{\underline{a}}^\alpha \overline{q}_\alpha^\infty) = \underline{0}$$.

Orthogonality (Ting, 1996) **for non-degenerate symmetries**

$$\underline{a}^\alpha \cdot \underline{b}^\beta + \underline{a}^\beta \cdot \underline{b}^\alpha = \delta_{\alpha\beta} = \overline{\underline{a}}^\alpha \cdot \overline{\underline{b}}^\beta + \overline{\underline{a}}^\beta \cdot \overline{\underline{b}}^\alpha$$
$$\underline{a}^\alpha \cdot \overline{\underline{b}}^\beta + \overline{\underline{a}}^\beta \cdot \underline{b}^\alpha = 0 = \overline{\underline{a}}^\alpha \cdot \underline{b}^\beta + \underline{a}^\beta \cdot \overline{\underline{b}}^\alpha$$

$$q_\alpha^\infty = \underline{a}^\alpha \cdot \underline{f}$$
$$\overline{q}_\alpha^\infty = \overline{\underline{a}}^\alpha \cdot \underline{f}$$

$$\underline{u}(\underline{x}) = \frac{1}{\pi}\Im\{\underline{a}^1 \otimes \underline{a}^1 \ln(z_1) + \underline{a}^2 \otimes \underline{a}^2 \ln(z_2)\} \cdot \underline{f}$$

$$\mathbf{G}(\underline{x}) = \frac{1}{\pi}\Im\{\underline{a}^1 \otimes \underline{a}^1 \ln(z_1) + \underline{a}^2 \otimes \underline{a}^2 \ln(z_2)\}$$

2D Barnett-Lothe integral formalism

- For degenerate symmetries, the proposed solution is incomplete. The displacement field needs to be adjusted (not done here).
- Alternatively, Barnett and Lothe (1973) developed a solution which remains valid irrespectively of the type of anisotropy:

$$2\underline{u}(r,\theta) = -\frac{1}{\pi}\ln(r)\mathbf{H}(\pi)\cdot\underline{f} - \mathbf{S}(\theta)\cdot\mathbf{H}(\pi)\cdot\underline{f} - \mathbf{H}(\theta)\cdot\mathbf{S}^T(\pi)\cdot\underline{f}$$

where the incomplete Barnett-Lothe integrals are

$$\mathbf{S}(\theta) = \frac{1}{\pi}\int_0^\theta \mathbf{N}^1(\psi)\mathrm{d}\psi \quad \text{and} \quad \mathbf{H}(\theta) = \frac{1}{\pi}\int_0^\theta \mathbf{N}^2(\psi)\mathrm{d}\psi \quad \text{where} \quad \begin{aligned}\mathbf{N}^1(\theta) &= -\mathbf{T}^{-1}(\theta)\cdot\mathbf{R}^T(\theta)\\ \mathbf{N}^2(\theta) &= \mathbf{T}^{-1}(\theta)\end{aligned}$$

with $R_{ik}(\theta) = L^0_{ijkl}n_j(\theta)m_l(\theta)$ and $T_{ik}(\theta) = L^0_{ijkl}m_j(\theta)m_l(\theta)$, while $\underline{n}(\theta) = \cos(\theta)\underline{e}_1 + \sin(\theta)\underline{e}_2$, $\underline{m}(\theta) = -\sin(\theta)\underline{e}_1 + \cos(\theta)\underline{e}_2$ so that

Active clockwise rotation of *n, ok?*

$$R_{ik}(\theta) = L^0_{i1k2}\cos^2(\theta) + (L^0_{i2k2} - L^0_{i1k1})\cos(\theta)\sin(\theta) - L^0_{i2k1}\sin^2(\theta)$$

$$T_{ik}(\theta) = L^0_{i2k2}\cos^2(\theta) - (L^0_{i1k2} + L^0_{i2k1})\cos(\theta)\sin(\theta) + L^0_{i1k1}\sin^2(\theta)$$

$$N^1_{ji}(\theta) \neq N^1_{ij}(\theta)$$
$$N^2_{ji}(\theta) = N^2_{ij}(\theta)$$
$$\Downarrow$$
$$\mathbf{H}^T = \mathbf{H}$$

- The 2D anisotropic Green functions then take the form

$$\boxed{2\mathbf{G}(r,\theta) = -\frac{1}{\pi}\ln(r)\mathbf{H}(\pi) - \mathbf{S}(\theta)\cdot\mathbf{H}(\pi) - \mathbf{H}(\theta)\cdot\mathbf{S}^T(\pi)} \; .$$

- Next, we find expressions for the incomplete Barnett-Lothe integrals in the case of specific material symmetries.

2D Barnett-Lothe integral formalism

- The gradients of the resulting Green functions

$$2G_{ij}(r,\theta) = -\frac{1}{\pi}\ln(r)H_{ij}(\pi) - S_{is}(\theta)H_{sj}(\pi) - H_{is}(\theta)S_{js}(\pi)$$

$$\partial_{x_{k_1}} f(r,\theta) = n_{k_1}(\theta)\partial_r f(r,\theta) + r^{-1} m_{k_1}(\theta)\partial_\theta f(r,\theta)$$

are obtained as follows, independently of material symmetries:

$$2G_{ij,k_1}(r,\theta) = -\frac{r^{-1}}{\pi}H_{ij}(\pi)n_{k_1}(\theta) - r^{-1}\partial_\theta[S_{is}(\theta)]H_{sj}(\pi)m_{k_1}(\theta) - r^{-1}\partial_\theta[H_{is}(\theta)]S_{js}(\pi)m_{k_1}(\theta)$$

$$2G_{ij,k_1}(r,\theta) = -\frac{r^{-1}}{\pi}\left[H_{ij}(\pi)n_{k_1}(\theta) + N^1_{is}(\theta)H_{sj}(\pi)m_{k_1}(\theta) + N^2_{is}(\theta)S_{js}(\pi)m_{k_1}(\theta)\right]$$

$$\pi\partial_\theta[S_{ij}(\theta)] = N^1_{ij}(\theta)$$

$$\pi\partial_\theta[H_{ij}(\theta)] = N^2_{ij}(\theta)$$

$$2G^{(1)}_{ij,k_1}(r,\theta) = g^1(r)h^1_{ijk_1}(\theta)$$

where

$$h^1_{ijk_1}(\theta) = H_{ij}(\pi)n_{k_1}(\theta) + N^1_{is}(\theta)H_{sj}(\pi)m_{k_1}(\theta) + N^2_{is}(\theta)S_{js}(\pi)m_{k_1}(\theta)$$

$$g^1(r) = -\frac{r^{-1}}{\pi}$$

so that

$$\partial_{k_2}[g^1(r)h^1_{ijk_1}(\theta)] = n_{k_2}(\theta)\partial_r[g^1(r)]h^1_{ijk_1}(\theta) + r^{-1}m_{k_2}(\theta)g^1(r)\partial_\theta[h^1_{ijk_1}(\theta)]$$

$$\partial_{k_2}[g^1(r)h^1_{ijk_1}(\theta)] = n_{k_2}(\theta)\pi^{-1}r^{-2}h^1_{ijk_1}(\theta) - r^{-1}m_{k_2}(\theta)\pi^{-1}r^{-1}\partial_\theta[h^1_{ijk_1}(\theta)]$$

$$\partial_{k_2}[g^1(r)h^1_{ijk_1}(\theta)] = \frac{r^{-2}}{\pi}\left[h^1_{ijk_1}(\theta)n_{k_2}(\theta) - \partial_\theta[h^1_{ijk_1}(\theta)]m_{k_2}(\theta)\right]$$

$$2G^{(2)}_{ij,k_1k_2}(r,\theta) = g^2(r)h^2_{ijk_1k_2}(\theta)$$

where

$$g^2(r) = \frac{r^{-2}}{\pi}$$

$$h^2_{ijk_1k_2}(\theta) = h^1_{ijk_1}(\theta)n_{k_2}(\theta) - \partial_\theta[h^1_{ijk_1}(\theta)]m_{k_2}(\theta)$$

2D Barnett-Lothe integral formalism

- Similarly, we have

$$\partial_{k_3}[g^2(r)h^2_{ijk_1k_2}(\theta)] = n_{k_3}(\theta)\partial_r[g^2(r)]h^2_{ijk_1k_2}(\theta) + r^{-1}m_{k_3}(\theta)g^2(r)\partial_\theta[h^2_{ijk_1k_2}(\theta)]$$

$$\partial_{k_3}[g^2(r)h^2_{ijk_1k_2}(\theta)] = -2n_{k_3}(\theta)\pi^{-1}r^{-3}h^2_{ijk_1k_2}(\theta) + r^{-1}m_{k_3}(\theta)\pi^{-1}r^{-2}\partial_\theta[h^2_{ijk_1k_2}(\theta)]$$

$$\partial_{k_3}[g^2(r)h^2_{ijk_1k_2}(\theta)] = -\frac{r^{-3}}{\pi}\left[2n_{k_3}(\theta)h^2_{ijk_1k_2}(\theta) - m_{k_3}(\theta)\partial_\theta[h^2_{ijk_1k_2}(\theta)]\right]$$

$$\boxed{2G^{(3)}_{ij,k_1k_2k_3}(r,\theta) = g^3(r)h^3_{ijk_1k_2k_3}(\theta)}$$

where

$$g^3(r) = -\frac{r^{-3}}{\pi}$$

$$h^3_{ijk_1k_2k_3}(\theta) = 2h^2_{ijk_1k_2}(\theta)n_{k_3}(\theta) - \partial_\theta[h^2_{ijk_1k_2}(\theta)]m_{k_3}(\theta)$$

- And

$$\partial_{k_4}[g^3(r)h^3_{ijk_1k_2k_3}(\theta)] = n_{k_4}(\theta)\partial_r[g^3(r)]h^3_{ijk_1k_2k_3}(\theta) + r^{-1}m_{k_4}(\theta)g^3(r)\partial_\theta[h^3_{ijk_1k_2k_3}(\theta)]$$

$$\partial_{k_4}[g^3(r)h^3_{ijk_1k_2k_3}(\theta)] = 3n_{k_4}(\theta)\pi^{-1}r^{-4}h^3_{ijk_1k_2k_3}(\theta) - r^{-1}m_{k_4}(\theta)\pi^{-1}r^{-3}\partial_\theta[h^3_{ijk_1k_2k_3}(\theta)]$$

$$\partial_{k_4}[g^3(r)h^3_{ijk_1k_2k_3}(\theta)] = \frac{r^{-4}}{\pi}\left[3n_{k_4}(\theta)h^3_{ijk_1k_2k_3}(\theta) - m_{k_4}(\theta)\partial_\theta[h^3_{ijk_1k_2k_3}(\theta)]\right]$$

$$\boxed{2G^{(4)}_{ij,k_1k_2k_3k_4}(r,\theta) = g^4(r)h^4_{ijk_1k_2k_3k_4}(\theta)}$$

where

$$g^4(r) = \frac{r^{-4}}{\pi}$$

$$h^4_{ijk_1k_2k_3k_4}(\theta) = 3h^3_{ijk_1k_2k_3}(\theta)n_{k_4}(\theta) - \partial_\theta[h^3_{ijk_1k_2k_3}(\theta)]m_{k_4}(\theta)$$

2D Barnett-Lothe integral formalism

- Again, $\partial_{k_5}[g^4(r)h^4_{ijk_1k_2k_3k_4}(\theta)] = n_{k_5}(\theta)\partial_r[g^4(r)]h^4_{ijk_1k_2k_3k_4}(\theta) + r^{-1}m_{k_5}(\theta)g^4(r)\partial_\theta[h^4_{ijk_1k_2k_3k_4}(\theta)]$

$$\partial_{k_5}[g^4(r)h^4_{ijk_1k_2k_3k_4}(\theta)] = -4n_{k_5}(\theta)\pi^{-1}r^{-5}h^4_{ijk_1k_2k_3k_4}(\theta) + r^{-1}m_{k_5}(\theta)\pi^{-1}r^{-4}\partial_\theta[h^4_{ijk_1k_2k_3k_4}(\theta)]$$

$$\partial_{k_5}[g^4(r)h^4_{ijk_1k_2k_3k_4}(\theta)] = -\frac{r^{-5}}{\pi}\left[3n_{k_5}(\theta)h^4_{ijk_1k_2k_3k_4}(\theta) - m_{k_5}(\theta)\partial_\theta[h^4_{ijk_1k_2k_3k_4}(\theta)]\right]$$

$$2G^{(5)}_{ij,k_1k_2k_3k_4k_5}(r,\theta) = g^5(r)h^5_{ijk_1k_2k_3k_4k_5}(\theta)$$

where

$$g^5(r) = -\frac{r^{-5}}{\pi}$$

$$h^5_{ijk_1k_2k_3k_4k_5}(\theta) = 4h^4_{ijk_1k_2k_3k_4}(\theta)n_{k_5}(\theta) - \partial_\theta[h^4_{ijk_1k_2k_3k_4}(\theta)]m_{k_5}(\theta)$$

- More generally, for $n \geq 1$, we have the following recurrence relations

$$2\pi G^{(n)}_{ij,k_1\ldots k_n}(r,\theta) = (-r)^{-n}h^n_{ijk_1\ldots k_n}(\theta)$$

$$h^n_{ijk_1\ldots k_n}(\theta) = (n-1)h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)n_{k_n}(\theta) - \partial_\theta[h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)]m_{k_n}(\theta) \quad \text{for } n \geq 2$$

$$\partial^k_\theta[h^n_{ijk_1\ldots k_n}(\theta)] = \sum_{s=0}^{k}\binom{k}{s}\left\{(n-1)\partial^{k-s}_\theta[h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)]\partial^s_\theta[n_{k_n}(\theta)] - \partial^{k-s+1}_\theta[h^{n-1}_{ijk_1\ldots k_{n-1}}(\theta)]\partial^{s+1}_\theta[n_{k_n}(\theta)]\right\}$$

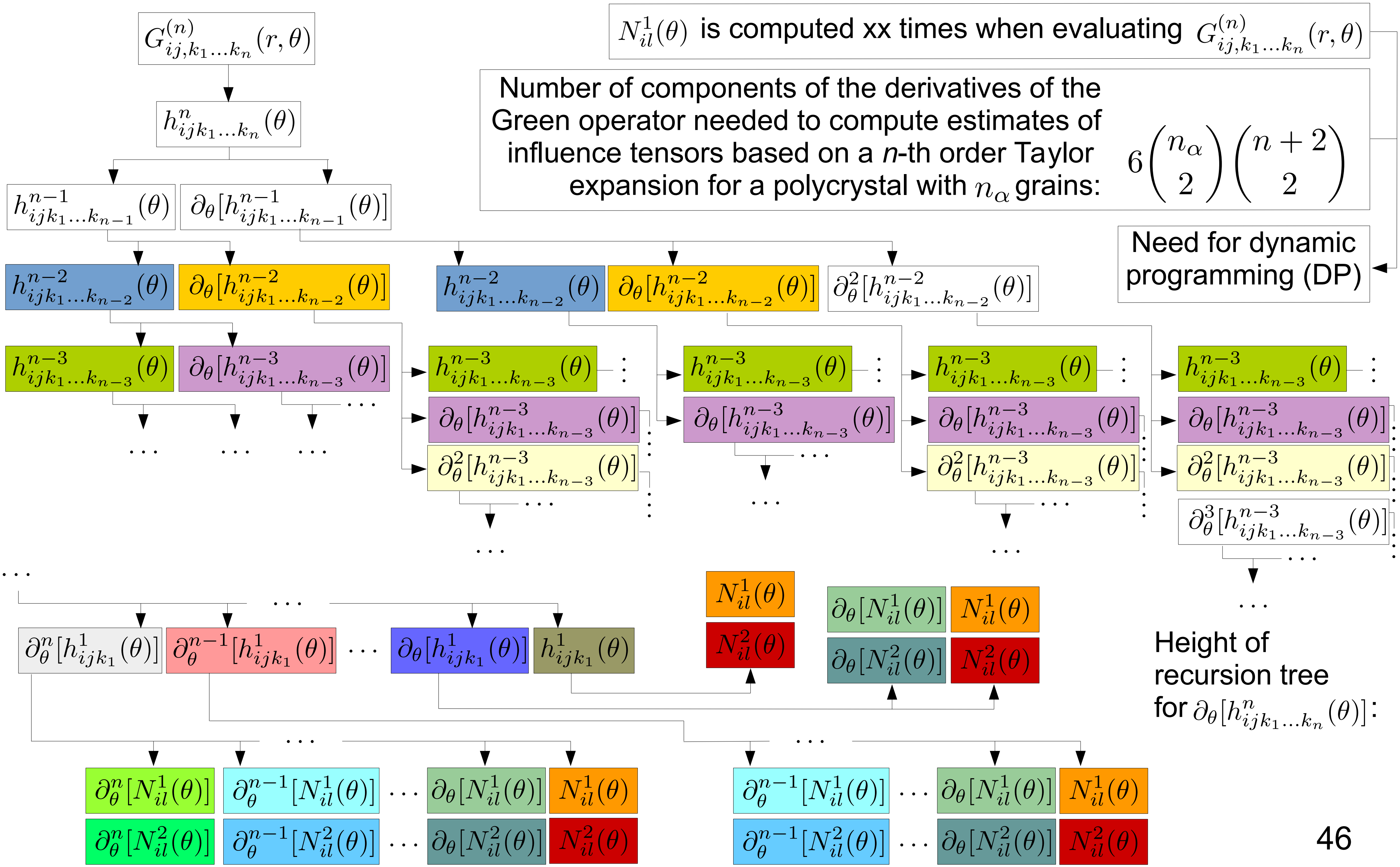
$$h^1_{ijk_1}(\theta) = H_{ij}n_{k_1}(\theta) + [N^1_{is}(\theta)H_{sj} + N^2_{is}(\theta)S_{js}]m_{k_1}(\theta)$$

$$\partial^k_\theta[h^1_{ijk_1}(\theta)] = H_{ij}\partial^k_\theta[n_{k_1}(\theta)] + \sum_{s=0}^{k}\binom{k}{s}\left\{H_{lj}\partial^{k-s}_\theta[N^1_{il}(\theta)] + S_{jl}\partial^{k-s}_\theta[N^2_{il}(\theta)]\right\}\partial^s_\theta[m_{k_1}(\theta)]$$

→ Requires evaluation of $\partial^k_\theta[N^1_{il}(\theta)]$ and $\partial^k_\theta[N^2_{il}(\theta)]$ for $k = 0, \ldots, n-1$

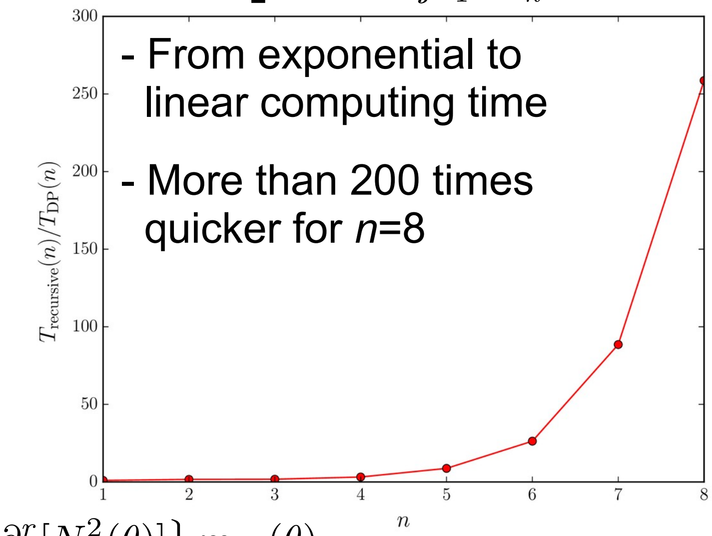
Drawback of a simple recursive implementation

- Computing the n-th derivative of an anisotropic Green's function at a location (r, θ) leads up to the following recurrence tree:



A bottom-up DP algorithm

- We derive the following bottom-up DP algorithm to compute $h^{n}_{ijk_1\ldots k_n}(\theta)$:



def $h^{n}_{ijk_1\ldots k_n}(\theta)$:

  $\texttt{d0hk} := \mathrm{zeros}(n)$

  for $k \in [1, n]$:

    for $rr \in [0, n-k]$:

      $r = n - k - rr$

      for $s \in [0, r]$:

        if $(s == 0)$:

          if $(k == 1)$:

            $\texttt{d0hk}[r+k-1] = H_{ij}\partial^{r}_{\theta}[n_{k_1}(\theta)] + \left\{H_{lj}\partial^{r}_{\theta}[N^{1}_{il}(\theta)] + S_{jl}\partial^{r}_{\theta}[N^{2}_{il}(\theta)]\right\} m_{k_1}(\theta)$

          else :

            $\texttt{d0hk}[r+k-1] = (k-1)\texttt{d0hk}[r+k-2]n_{k_k}(\theta) - \texttt{d0hk}[r+k-1]\partial^{1}_{\theta}[n_{k_k}(\theta)]$

        else :

          if $(k == 1)$:

            $\texttt{d0hk}[r+k-1] += \binom{r}{s} \left\{H_{lj}\partial^{r-s}_{\theta}[N^{1}_{il}(\theta)] + S_{jl}\partial^{r-s}_{\theta}[N^{2}_{il}(\theta)]\right\} \partial^{s}_{\theta}[m_{k_1}(\theta)]$

          else :

            $\texttt{d0hk}[r+k-1] += \binom{r}{s} \left\{(k-1)\texttt{d0hk}[r-s+k-2]\partial^{s}_{\theta}[n_{k_k}(\theta)] - \texttt{d0hk}[r-s+k-1]\partial^{s+1}_{\theta}[n_{k_k}(\theta)]\right\}$

    #At this stage, $r \in [0, n-k] \implies \texttt{d0hk}[r+k-1] = \partial^{r}_{\theta}[h^{k}_{ijk_1\ldots k_k}(\theta)]$

  #At this stage, $k \in [1, n] \implies \texttt{d0hk}[k-1] = h^{k}_{ijk_1\ldots k_k}(\theta)$

  return $\texttt{d0hk}[n-1]$

2D Anisotropy

- Polar representation of 2D anisotropic stiffnesses, see Vannucci (2016)

$$
\begin{aligned}
L_{1111} &= T_0 + 2T_1 + R_0\cos(4\Phi_0) + 4R_1\cos(2\Phi_1)\\
L_{1112} &= R_0\sin(4\Phi_0) + 2R_1\sin(2\Phi_1)\\
L_{1122} &= -T_0 + 2T_1 - R_0\cos(4\Phi_0)\\
L_{1212} &= T_0 - R_0\cos(4\Phi_0)\\
L_{2212} &= -R_0\sin(4\Phi_0) + 2R_1\sin(2\Phi_1)\\
L_{2222} &= T_0 + 2T_1 + R_0\cos(4\Phi_0) - 4R_1\cos(2\Phi_1)
\end{aligned}
$$

$T_0,\ T_1$: Isotropic polar invariants

$R_0,\ R_1,\ \Phi_0-\Phi_1$: Anisotropic polar invariants

Substitute Φ_j by $\Phi_j - \theta$ for counter clockwise positive passive rotation

Conditions for positive strain energy

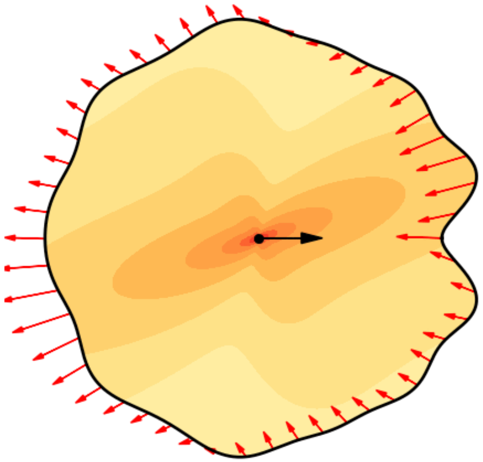
$$
\begin{aligned}
T_0 - R_0 &> 0,\\
T_1(T_0^2 - R_0^2) - 2R_1^2\{T_0 - R_0\cos[4(\Phi_0-\Phi_1)]\} &> 0,\\
R_0 &\geq 0,\\
R_1 &\geq 0.
\end{aligned}
$$

$$R_0R_1^2\sin[4(\Phi_0-\Phi_1)] \neq 0$$

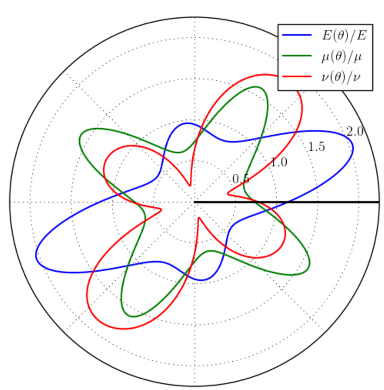
$$R_0R_1^2\sin[4(\Phi_0-\Phi_1)] = 0 \implies \text{Symmetry}$$

Validation

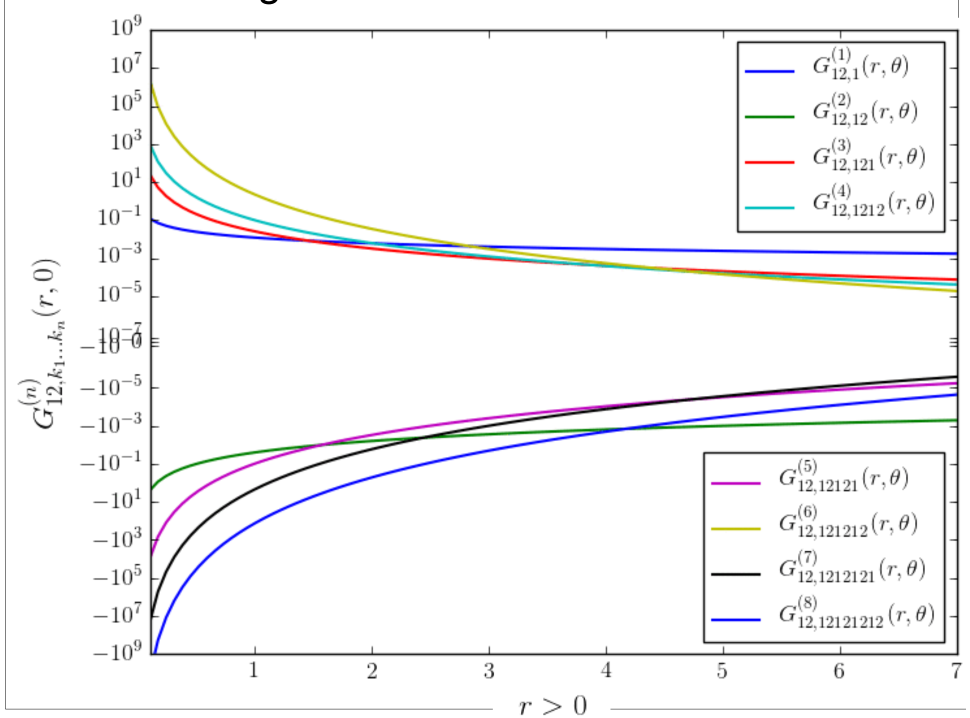
Equilibrated traction fields on random curves



Polar diagram of generalized moduli



Computed components of some gradients of the Green's function



2D Orthotropy

- Polar representation of 2D orthotropic stiffnesses, see Vannucci (2016)

$$
\begin{aligned}
L_{1111} &= (-1)^K R_0 \cos(4\theta) + 4R_1 \cos(2\theta) + T_0 + 2T_1, \\
L_{1112} &= -(-1)^K R_0 \sin(4\theta) - 2R_1 \sin(2\theta), \\
L_{1122} &= -(-1)^K R_0 \cos(4\theta) - T_0 + 2T_1, \\
L_{1212} &= T_0 - (-1)^K R_0 \cos(4\theta), \\
L_{2212} &= (-1)^K R_0 \sin(4\theta) - 2R_1 \sin(2\theta), \\
L_{2222} &= (-1)^K R_0 \cos(4\theta) - 4R_1 \cos(2\theta) + T_0 + 2T_1
\end{aligned}
$$

$$\Downarrow \quad S_{22} = -S_{11} \quad \Downarrow \quad \mathrm{tr}\mathbf{S} = 0$$

$T_0,\ T_1$: Isotropic polar invariants

$R_0,\ R_1,\ K$: Anisotropic polar invariants, with $K = \pm 1$

θ is a counter – clockwise positive passive rotation

Conditions for positive strain energy

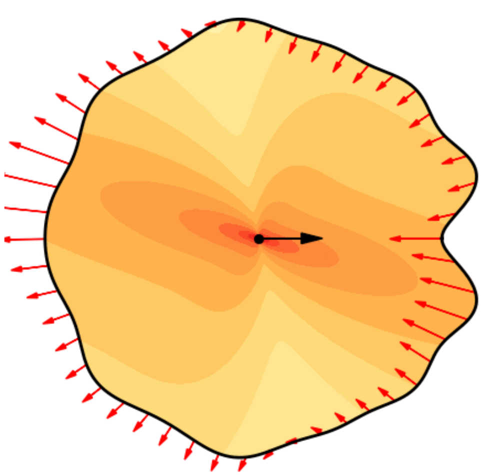
$$
\begin{aligned}
T_0 - R_0 &> 0, \\
T_1[T_0^2 + (-1)^K R_0] - 2R_1^2 &> 0, \\
R_0 &\geq 0, \\
R_1 &\geq 0.
\end{aligned}
$$

$$\sin[4(\Phi_0 - \Phi_1)] = 0$$

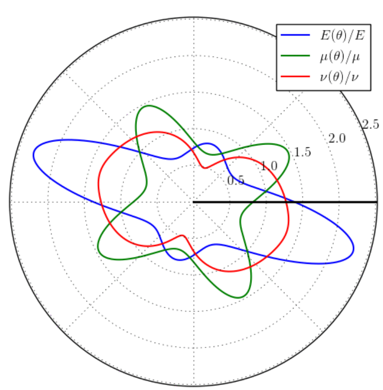
$$
\begin{aligned}
R_0 = 0 &\implies R_0 - \text{orthotropy} \\
R_0 = 0 \text{ and } R_1 = 0 &\implies \text{Isotropy}
\end{aligned}
$$

Validation

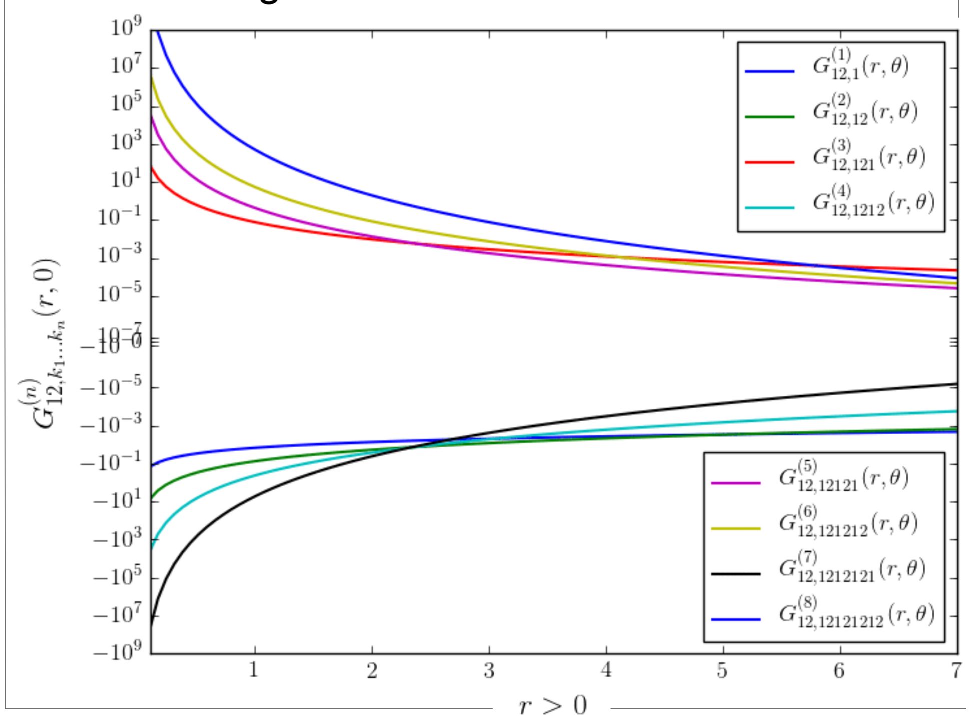
Equilibrated traction fields on random curves



Polar diagram of generalized moduli



Computed components of some gradients of the Green's function



2D R0-orthotropy

- Polar representation of 2D R0-orthotropic stiffnesses (Vannucci, 2016)

$$
\begin{aligned}
L_{1111} &= 4R_1\cos(2\theta) + T_0 + 2T_1,\\
L_{1112} &= -2R_1\sin(2\theta),\\
L_{1122} &= -T_0 + 2T_1,\\
L_{1212} &= T_0,\\
L_{2212} &= -2R_1\sin(2\theta),\\
L_{2222} &= -4R_1\cos(2\theta) + T_0 + 2T_1
\end{aligned}
$$

$$\Downarrow$$
$$S_{22} = -S_{11}$$
$$\Downarrow$$
$$\mathrm{tr}\mathbf{S} = 0$$

Conditions for positive strain energy

$$
\begin{aligned}
T_0 &> 0,\\
T_1 T_0^2 - 2R_1^2 &> 0,\\
R_1 &\geq 0.
\end{aligned}
$$

$\sin[4(\Phi_0 - \Phi_1)] = 0$ and $R_0 = 0$

$R_1 = 0 \implies$ Isotropy

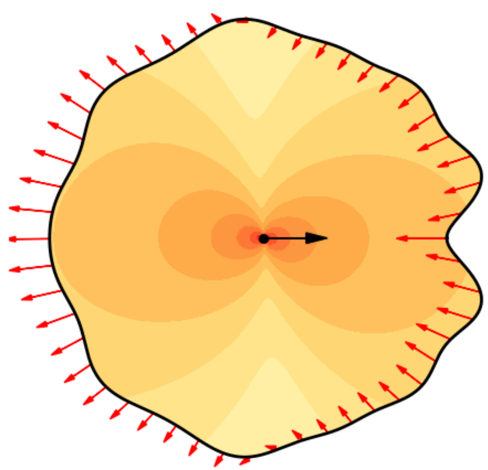
$T_0,\ T_1$: Isotropic polar invariants

R_1 : Anisotropic polar invariant

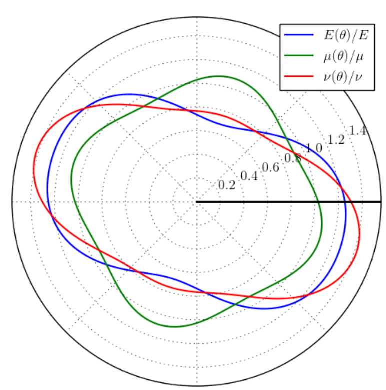
θ is a counter – clockwise positive passive rotation

Validation

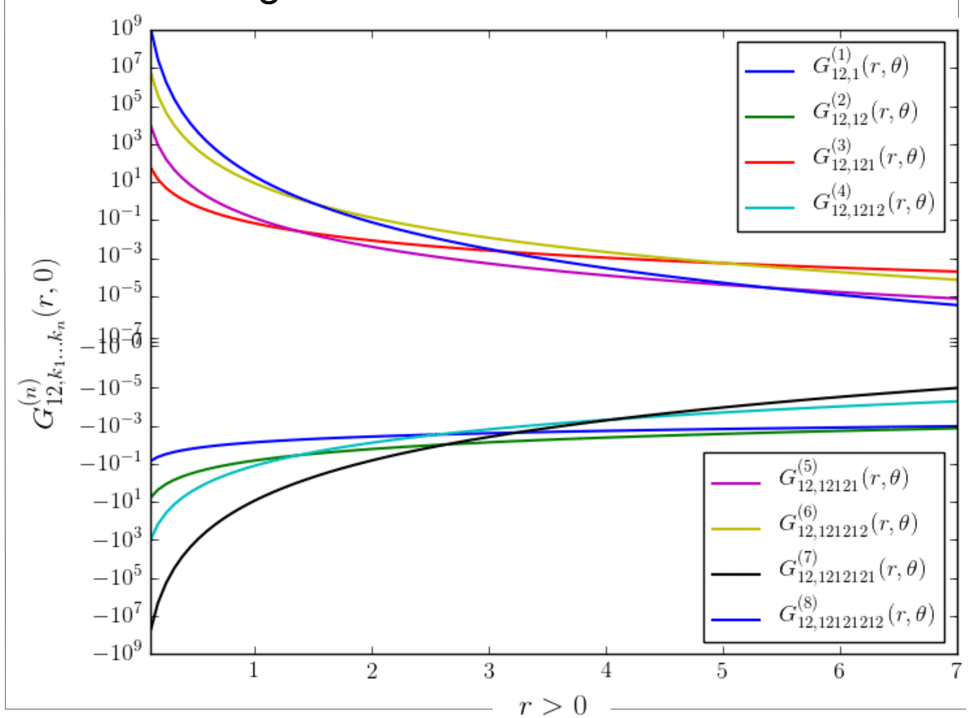
Equilibrated traction fields on random curves



Polar diagram of generalized moduli



Computed components of some gradients of the Green's function



2D square symmetry

- Polar representation of 2D square symmetry, see Vannucci (2016)

$$L_{1111} = T_0 + 2T_1 + R_0 \cos(4\theta),$$
$$L_{1112} = -R_0 \sin(4\theta),$$
$$L_{1122} = -T_0 + 2T_1 - R_0 \cos(4\theta),$$
$$L_{1212} = T_0 - R_0 \cos(4\theta),$$
$$L_{2212} = R_0 \sin(4\theta),$$
$$L_{2222} = T_0 + 2T_1 + R_0 \cos(4\theta)$$

$\Downarrow$

$$S_{11} = S_{22} = 0$$
$$S_{21} = -S_{12}$$
$$\Downarrow$$
$$\mathrm{skew}\mathbf{S} = \mathbf{S}$$

$\Downarrow$

$$H_{11} = H_{22}$$

Conditions for positive strain energy

$$T_0 - R_0 > 0$$
$$T_1(T_0^2 - R_0^2) - 2R_1^2\{T_0 - R_0 \cos[4(\Phi_0 - \Phi_1)]\} > 0$$
$$R_0 \geq 0$$
$$R_1 \geq 0.$$

$$R_1 = 0$$

$$R_0 = 0 \implies \text{Isotropy}$$

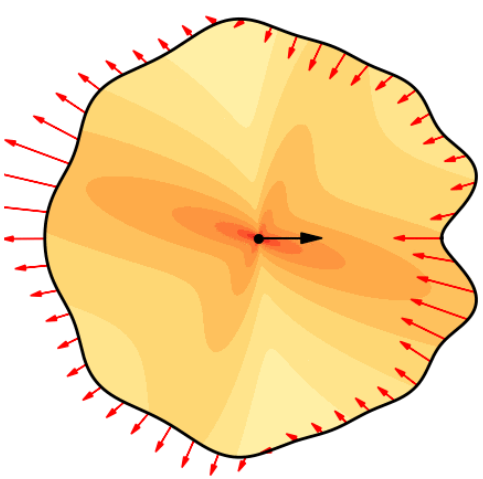
$T_0,\ T_1$: Isotropic polar invariants

R_0 : Anisotropic polar invariant

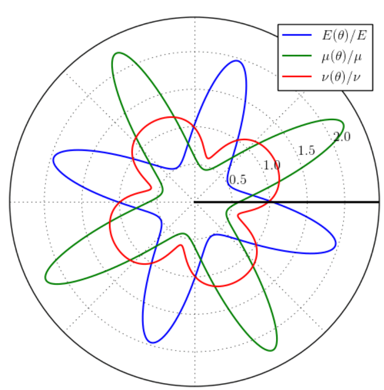
θ is a counter – clockwise positive passive rotation

Validation

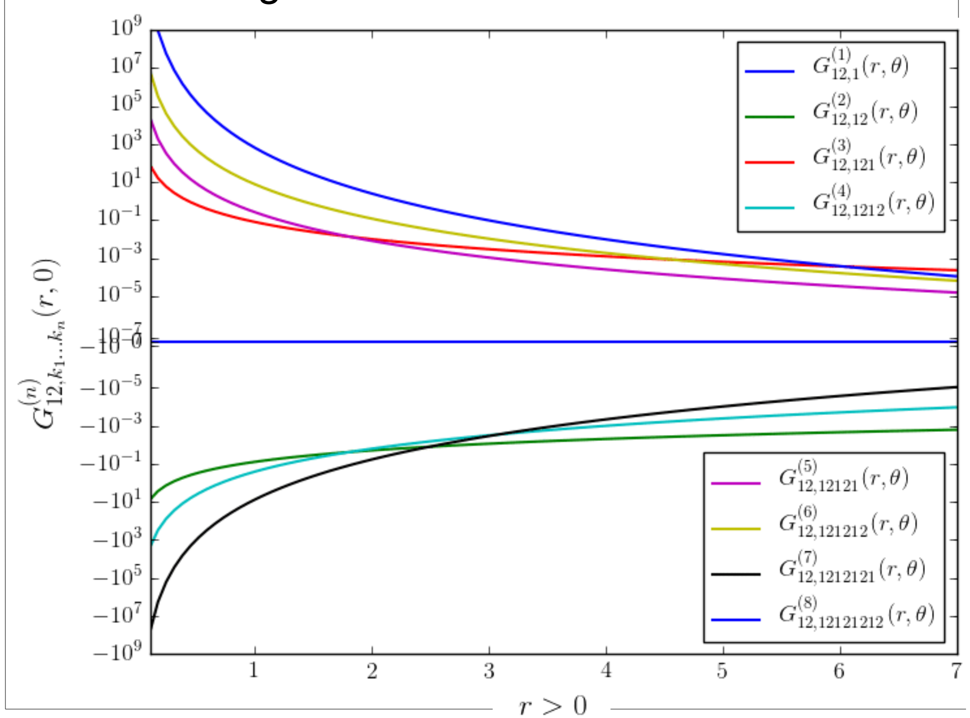
Equilibrated traction fields on random curves



Polar diagram of generalized moduli



Computed components of some gradients of the Green's function



2D Isotropy

- Polar representation of 2D anisotropic stiffnesses, see Vannucci (2016)

$$
\begin{aligned}
L_{1111} &= T_0 + 2T_1,\\
L_{1112} &= 0,\\
L_{1122} &= -T_0 + 2T_1,\\
L_{1212} &= T_0,\\
L_{2212} &= 0\\
L_{2222} &= T_0 + 2T_1
\end{aligned}
$$

$\Downarrow$

$$
\begin{aligned}
S_{11} &= S_{22} = 0\\
S_{12} &= -\frac{\mu_{2D}}{\kappa_{2D} + \mu_{2D}}\\
S_{21} &= -S_{12}
\end{aligned}
$$

$\Downarrow$

$$
\begin{aligned}
H_{11} &= \frac{\kappa_{2D} + 2\mu_{2D}}{2\mu_{2D}(\kappa_{2D} + \mu_{2D})}\\
H_{22} &= \frac{\kappa_{2D} + 2\mu_{2D}}{2\mu_{2D}(\kappa_{2D} + \mu_{2D})}\\
H_{12} &= H_{21} = 0
\end{aligned}
$$

Conditions for positive strain energy

$$
T_0 > 0,\quad T_1 > 0.
$$

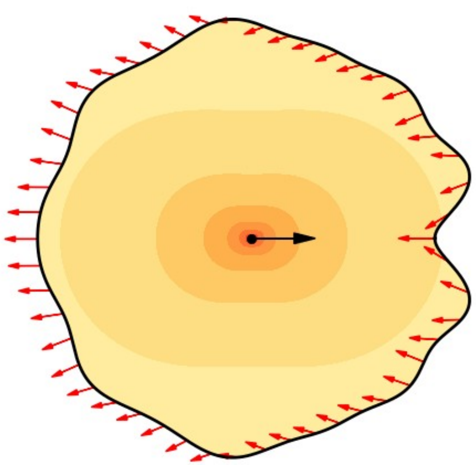
$R_0 = 0$ and $R_1 = 0$

$T_0,\ T_1$: Isotropic polar invariants

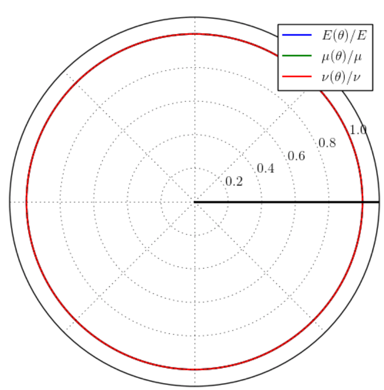
$\kappa_{2D},\ \mu_{2D}$: Bulk and shear moduli

$$
T_0 = \mu_{2D}\qquad 2T_1 = \kappa_{2D}
$$

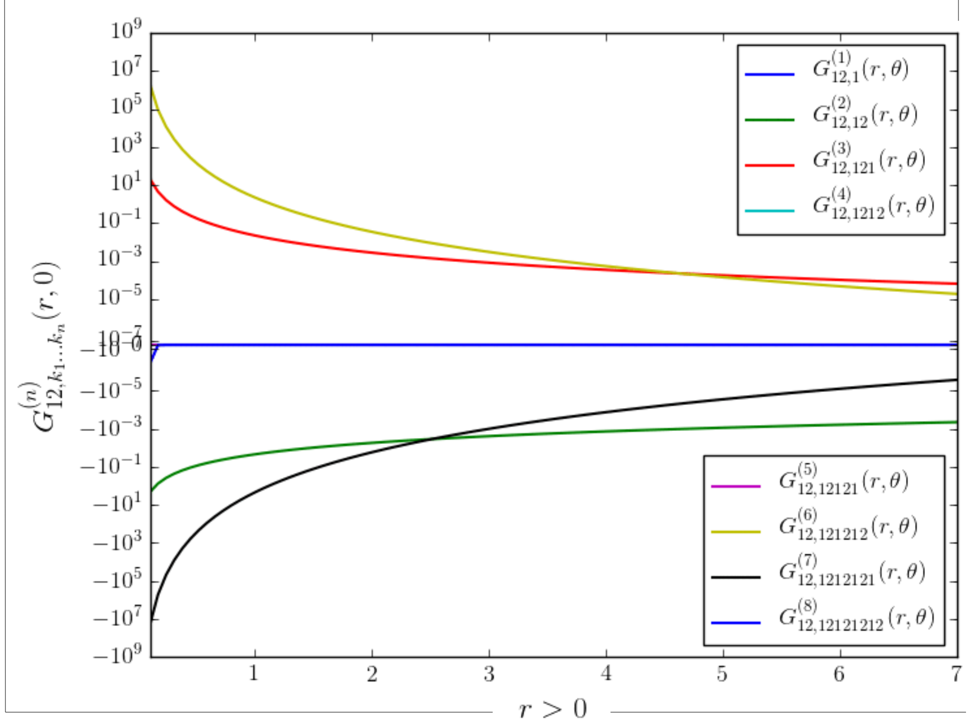
Validation
Equilibrated traction fields on random curves



Polar diagram of generalized moduli



Computed components of some gradients of the Green's function



Green operator for strains

- So far, we computed gradients of the Green's function away from the origin, i.e. with *r>0*. By continuity, we have

$$G^{(n)}_{ij,k_1\dots k_n}(r,\theta) = G^{(n)}_{ij,k'_1\dots k'_n}(r,\theta) \ \text{ for } r>0$$

  for every permutation $(k'_1 \dots k'_n)$ of $(k_1 \dots k_n)$.

- The "Green operator for strain" is then defined by

$$4\Gamma_{ijkl}(r,\theta) := G^{(2)}_{ik,jl}(r,\theta) + G^{(2)}_{il,jk}(r,\theta) + G^{(2)}_{jk,il}(r,\theta) + G^{(2)}_{jl,ik}(r,\theta)$$

  so that Γ_{ijkl} is minor and major symmetric.

- The gradients/derivatives of the operator are then given by

$$4\Gamma^{(n)}_{ijkl,k_1\dots k_n}(r,\theta) = G^{(n+2)}_{ik,jlk_1\dots k_n}(r,\theta) + G^{(n+2)}_{il,jkk_1\dots k_n}(r,\theta) + G^{(n+2)}_{jk,ilk_1\dots k_n}(r,\theta) + G^{(n+2)}_{jl,ikk_1\dots k_n}(r,\theta)$$

- Consequently, for *r>0*, we have
  - $\Gamma^{(n)}_{ijkl,k_1\dots k_n}(r,\theta) = \Gamma^{(n)}_{ijkl,k'_1\dots k'_n}(r,\theta)$ for every permutation $(k'_1 \dots k'_n)$ of $(k_1 \dots k_n)$,
  - $\Gamma^{(n)}_{ijkl,k_1\dots k_n}(r,\theta) = \Gamma^{(n)}_{klij,k_1\dots k_n}(r,\theta)$ and
  - $\Gamma^{(n)}_{ijkl,k_1\dots k_n}(r,\theta) = \Gamma^{(n)}_{jikl,k_1\dots k_n}(r,\theta) = \Gamma^{(n)}_{jilk,k_1\dots k_n}(r,\theta) = \Gamma^{(n)}_{ijlk,k_1\dots k_n}(r,\theta)$.
- Also, we recall that $\Gamma^{(n)}_{ijkl,k_1\dots k_n}(\underline{x}_{\gamma\alpha}) = (-1)^k \Gamma^{(n)}_{ijkl,k_1\dots k_n}(\underline{x}_{\alpha\gamma})$.
- Given those symmetries, we want to minimize the amount of computation

Table of gradient components of Green operators

- For some given n, we need to compute

$$\Gamma_{ijkl,k_1}(\underline{x}_{\gamma\alpha}),\ \Gamma^{(1)}_{ijkl,k_1}(\underline{x}_{\gamma\alpha}),\ \Gamma^{(2)}_{ijkl,k_1k_2}(\underline{x}_{\gamma\alpha}),\ \ldots,\Gamma^{(n)}_{ijkl,k_1\ldots k_k}(\underline{x}_{\gamma\alpha})$$

  for every pair $(\Omega_\alpha, \Omega_\gamma)$ of grains with $\alpha \neq \gamma$.

$\underline{x}_{\gamma\alpha} := \underline{x}_\alpha - \underline{x}_\gamma$

- For all $(\Omega_\alpha, \Omega_\gamma)$ such that $\alpha < \gamma$:
  - For all $ijkl \in \{1111, 1122, 1112, 2222, 2212, 1212\}$:
    - For all $k \in [0, n]$:
      - For all $i_1 \in [0, k]$:
        - » Compute $\mathrm{d}\Gamma[\alpha][\gamma][i_{ijkl}][k][i_1] := \Gamma^{(k)}_{ijkl,\overbrace{11\ldots1}^{(i_1\ \text{times})}\overbrace{22\ldots2}^{(k-i_1\ \text{times})}}$

Adjust for periodicity

- All necessary components of the derivatives can be obtained by symmetry from the values stored in $\mathrm{d}\Gamma$.

if $\alpha > \gamma$:

$$\Gamma^{(k)}_{ijkl\overbrace{11\ldots1}^{(i_1\ \text{times})}\overbrace{22\ldots2}^{(k-i_1\ \text{times})}}(\underline{x}_{\gamma\alpha}) = (-1)^k \mathrm{d}\Gamma[\gamma][\alpha][i_{ijkl}][k][i_1]$$

```
min_diff(x_gamma, x_alpha, L):

   dx = x_alpha - x_gamma

   if (dx[0] > L/2):
      dx[0] = -(L - dx[0])
   else if (dx[0] < -L/2):
      dx[0] = L+dx[0]

   if (dx[1] > L/2):
      dx[1] = -(L - dx[1])
   else if (dx[1] < -L/2.)
      dx[1] = L+dx[1]

   return dx;
```

- Number of components to compute:

$$6\binom{n_\alpha}{2}\binom{n+2}{2} = \frac{3(n_\alpha-1)n_\alpha(n+1)(n+2)}{2}$$

Q: For some fixed *n*, can we take less interactions into account i.e. compute influence tensors based on some $\tilde{n}_\alpha < n_\alpha$?

Idea of "<u>*k*-fold neighborhoods</u>"

Base case for verification and validation

- As a first application, we consider a 2D periodic array of anisotropic squares. The corresponding Minkowski tensors of interest have components

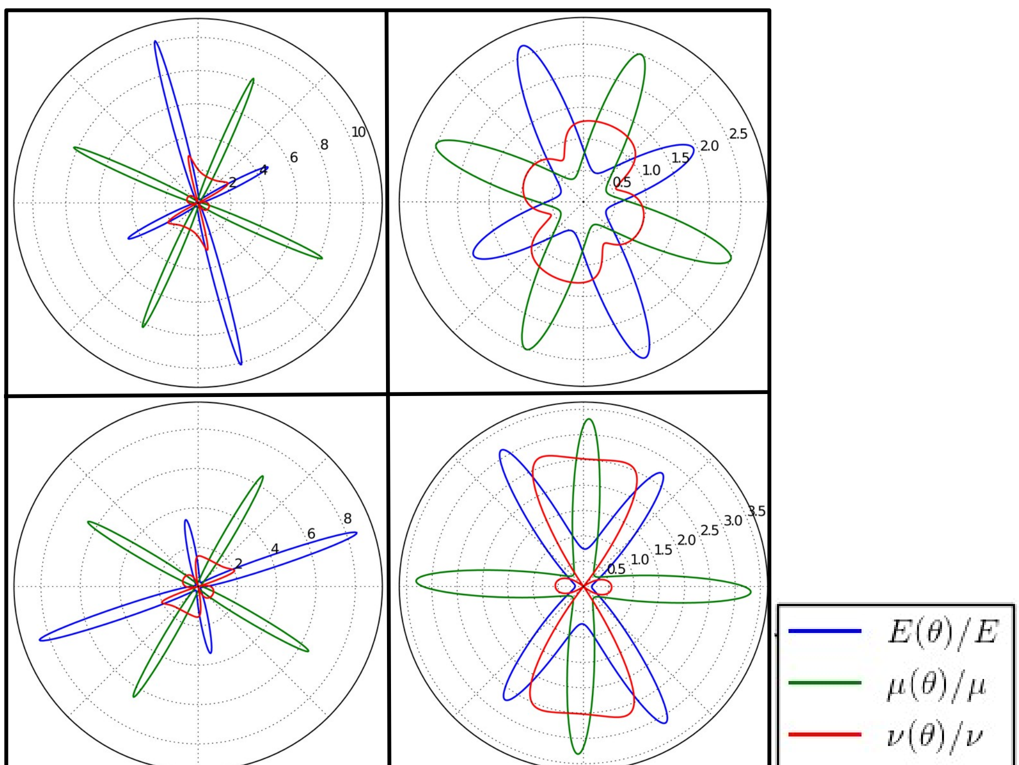
$$[\mathcal{W}_0^{r,0}](n_1) := [\mathcal{W}_0^{r,0}]_{\underbrace{11\ldots1}_{(n_1 \text{ times})}\underbrace{22\ldots2}_{(r-n_1 \text{ times})}}$$

$n_1 \in [0, r]$

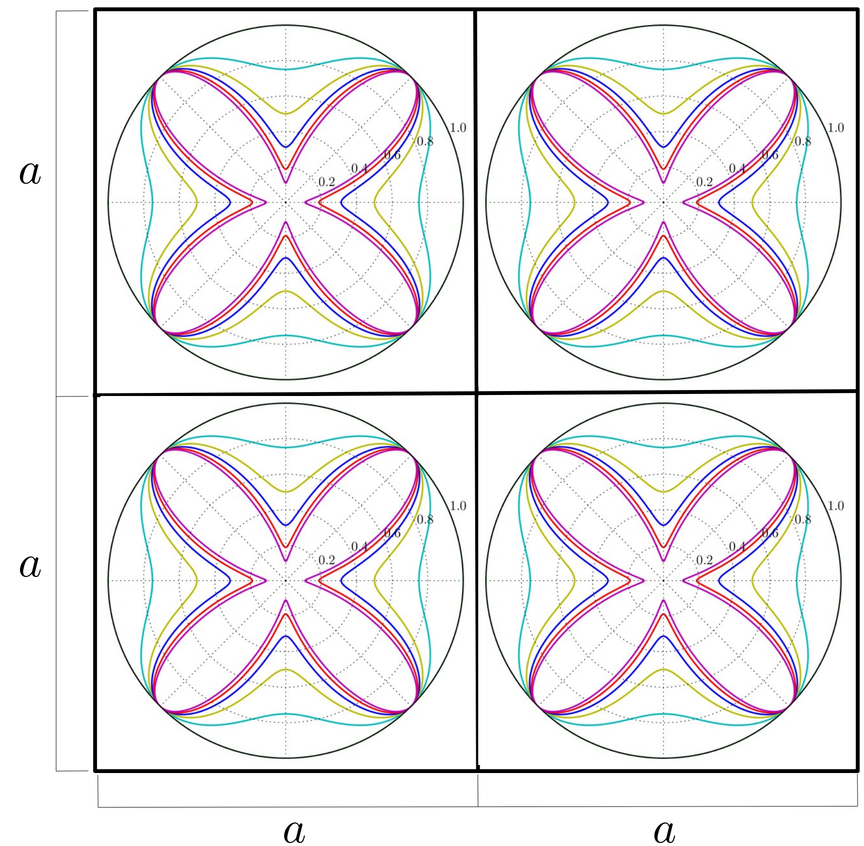
$n_2 := r - n_1$

$$[\mathcal{W}_0^{r,0}](n_1) = \frac{(a/2)^{n_1+n_2+2} - (-a/2)^{n_1+1}(a/2)^{n_2+1} - (a/2)^{n_1+1}(-a/2)^{n_2+1} + (-a/2)^{n_1+1}(-a/2)^{n_2+1}}{(n_1+1)(n_2+1)}$$

Polar diagram of generalized moduli



Reynolds glyphs of normalized Minkowski tensors $\mathcal{W}_0^{r,0}$ for $r \leq 12$



Extra-computation required for the evaluation of self-influence tensors

- The computation of the components $({}^{n}T^{\alpha\alpha}_{0,0})_{ijkl}$ requires to know $\mathcal{W}_0^{i,0}(\Omega^{\gamma}_{\alpha})$ for *i=0, ..., n* for some fixed $\gamma \neq \alpha$. We have

$$\mathcal{W}_0^{i,0}(\Omega^{\gamma}_{\alpha}) = \sum_{t=0}^{i} \binom{i}{t} \underline{x}_{\gamma\alpha}^{\otimes t} \odot \mathcal{W}_0^{i-t,0}(\Omega'_{\alpha})$$

where

$$[\mathcal{W}_0^{i,0}(\Omega'_{\alpha})](n_1) := [\mathcal{W}_0^{i,0}(\Omega'_{\alpha})]_{\overbrace{(n_1 \text{ times})}^{11\ldots1}\overbrace{(i-n_1 \text{ times})}^{22\ldots2}}$$

$$[\underline{x}_{\gamma\alpha}^{\otimes t} \odot \mathcal{W}_0^{i-t,0}(\Omega'_{\alpha})](n_1) := [\underline{x}_{\gamma\alpha}^{\otimes t} \odot \mathcal{W}_0^{i-t,0}(\Omega'_{\alpha})]_{\overbrace{(n_1 \text{ times})}^{11\ldots1}\overbrace{(i-n_1 \text{ times})}^{22\ldots2}}$$

$n_1 \in [0, i]$

$$[\underline{x}_{\gamma\alpha}^{\otimes t} \odot \mathcal{W}_0^{i-t,0}(\Omega'_{\alpha})](n_1) = \binom{i}{n_1}^{-1} \sum_{k=\max\{0,n_1-i+t\}}^{\min\{t,n_1\}} \binom{t}{k}\binom{i-t}{n_1-k} (x_1^{\gamma\alpha})^k (x_2^{\gamma\alpha})^{t-k} [\mathcal{W}_0^{i-t,0}(\Omega'_{\alpha})](n_1-k)$$

- Similarly, the computation of the components $({}^{n}T^{\alpha\alpha}_{r,s})_{r_1\ldots r_r ijkl s_1\ldots s_s}$ requires to know ${}^{\gamma}\widetilde{\mathcal{W}}_0^{i|s,0}(\Omega'_{\alpha})$ for *s=0, ..., p* and *i=0, ..., n* with some fixed $\gamma \neq \alpha$.
We have

$${}^{\gamma}\widetilde{\mathcal{W}}_0^{i|s,0}(\Omega'_{\alpha}) = \sum_{t=0}^{i} \binom{i}{t} (\underline{x}_{\gamma\alpha})^{\otimes i-t} \overset{i-t,t}{\odot} \mathcal{W}_0^{t+s,0}(\Omega'_{\alpha})$$

where

$$[(\underline{x}_{\gamma\alpha})^{\otimes i-t} \overset{i-t,t}{\odot} \mathcal{W}_0^{t+s,0}(\Omega'_{\alpha})](n_1, n_{s_1}) := [(\underline{x}_{\gamma\alpha})^{\otimes i-t} \overset{i-t,t}{\odot} \mathcal{W}_0^{t+s,0}(\Omega'_{\alpha})]_{\overbrace{(n_1 \text{ times})}^{11\ldots1}\overbrace{(i-n_1 \text{ times})}^{22\ldots2}\overbrace{(n_{s_1} \text{ times})}^{11\ldots1}\overbrace{(s-n_{s_1} \text{ times})}^{22\ldots2}}$$

$$[(\underline{x}_{\gamma\alpha})^{\otimes i-t} \overset{i-t,t}{\odot} \mathcal{W}_0^{t+s,0}(\Omega'_{\alpha})](n_1, n_{s_1}) = \binom{i}{n_1}^{-1} \sum_{q=\max\{0,n_1-t\}}^{\min\{i-t,n_1\}} \binom{i-t}{q}\binom{t}{n_1-q} (x_1^{\gamma\alpha})^q (x_2^{\gamma\alpha})^{i-t-q} [\mathcal{W}_0^{t+s,0}(\Omega'_{\alpha})](n_1-q+n_{s_1})$$

$n_1 \in [0, i]$

Post-processing

- Once an estimate of the polarization stress field is obtained, there are different ways to obtain the corresponding strain field
  - First, from the very definition of the polarization, we have

$$\boldsymbol{\varepsilon}(\underline{x}) = \Delta\mathbb{M}(\underline{x}) : \boldsymbol{\tau}(\underline{x})$$

    If so, we can recover closed form expressions of the corresponding piecewise polynomial strain and strain fields:

$$\boldsymbol{\varepsilon}^{h_p}(\underline{x}) = \sum_{\alpha} \left(\chi_\alpha(\underline{x})\boldsymbol{\varepsilon}^\alpha + \chi_\alpha(\underline{x}) \sum_{k=1}^{p} \left\langle \boldsymbol{\varepsilon}^\alpha \boldsymbol{\partial}^k, (\underline{x} - \underline{x}^\alpha)^{\otimes^k} \right\rangle_k \right)$$

    and $$\boldsymbol{\sigma}^{h_p}(\underline{x}) = \sum_{\alpha} \left(\chi_\alpha(\underline{x})\boldsymbol{\sigma}^\alpha + \chi_\alpha(\underline{x}) \sum_{k=1}^{p} \left\langle \boldsymbol{\sigma}^\alpha \boldsymbol{\partial}^k, (\underline{x} - \underline{x}^\alpha)^{\otimes^k} \right\rangle_k \right)$$

    However, as we do so, we note that *the "prescribed" mean strain state is not recovered*.

  - Another possibility is to exploit the following form of the Lippman-Schwinger equation

$$\boldsymbol{\varepsilon}(\underline{x}) = \overline{\boldsymbol{\varepsilon}} - {}^{n}\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_p}(\underline{x})$$

    for which derivations as the ones carried over for the definition of the influence tensors is needed.

Post-processing

- Once an estimate of the polarization stress field is obtained, there are different ways to obtain the corresponding strain field

  – First, from the very definition of the polarization, we have

$$\boldsymbol{\varepsilon}(\underline{x}) = \Delta\mathbb{M}(\underline{x}) : \boldsymbol{\tau}(\underline{x})$$

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  – Another possibility is to exploit the following form of the Lippman-Schwinger equation

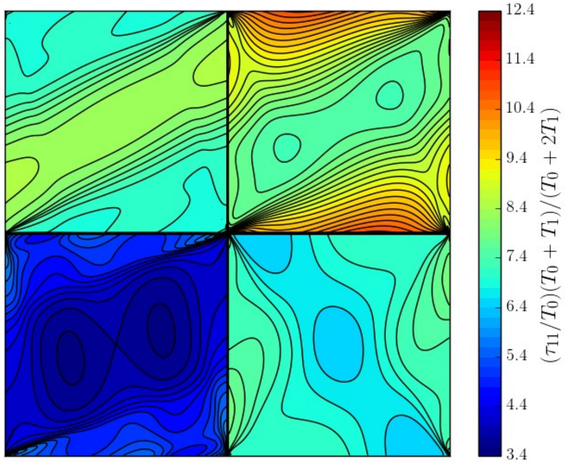
$$\boldsymbol{\varepsilon}(\underline{x}) = \overline{\boldsymbol{\varepsilon}} - {}^{n}\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_p}$$

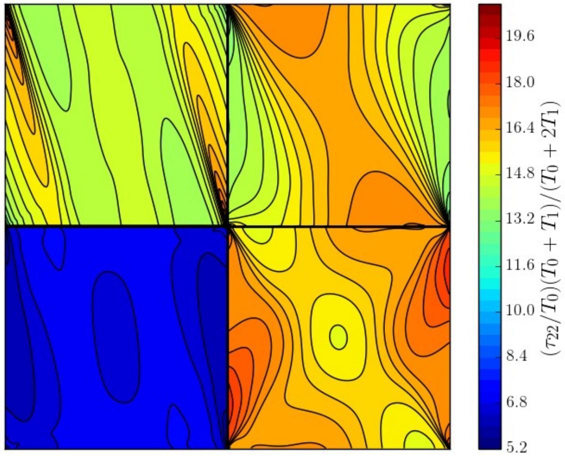
Work in progress

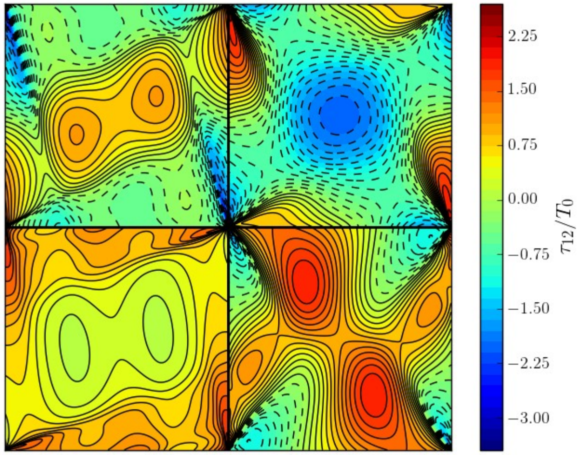
  for which derivations as the ones carried on of the influence tensors is needed.

Results

- Uniaxial average strain, $\langle \boldsymbol{\varepsilon} \rangle = \underline{e}_2 \otimes \underline{e}_2$





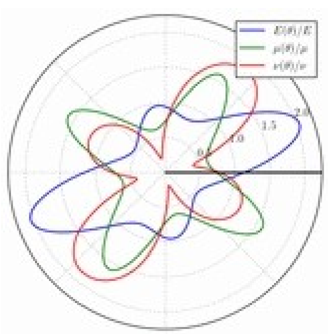


Fixing the method

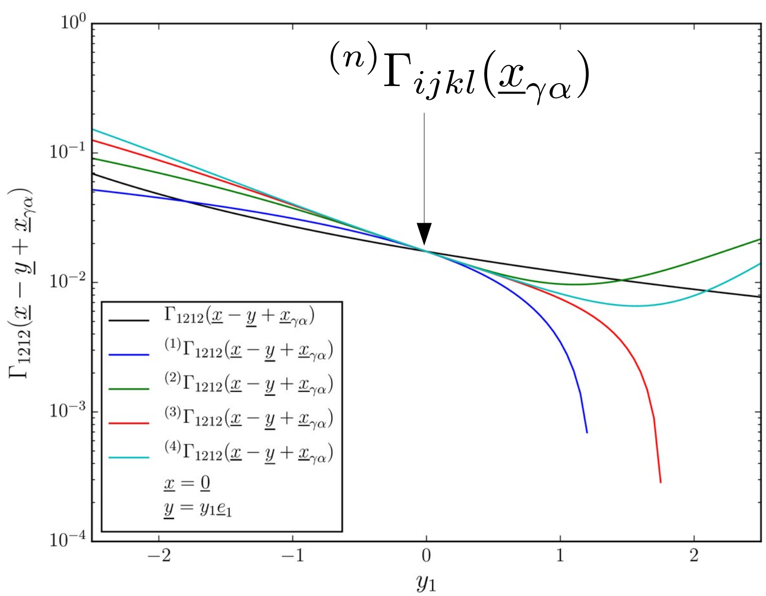
- Currently, the method does not work.
- Possible sources of error:
  - Inaccuracy of the Taylor expansion of the Green operator for strains.
  - Singularity in the integral equations for the influence tensors are not taken into account.

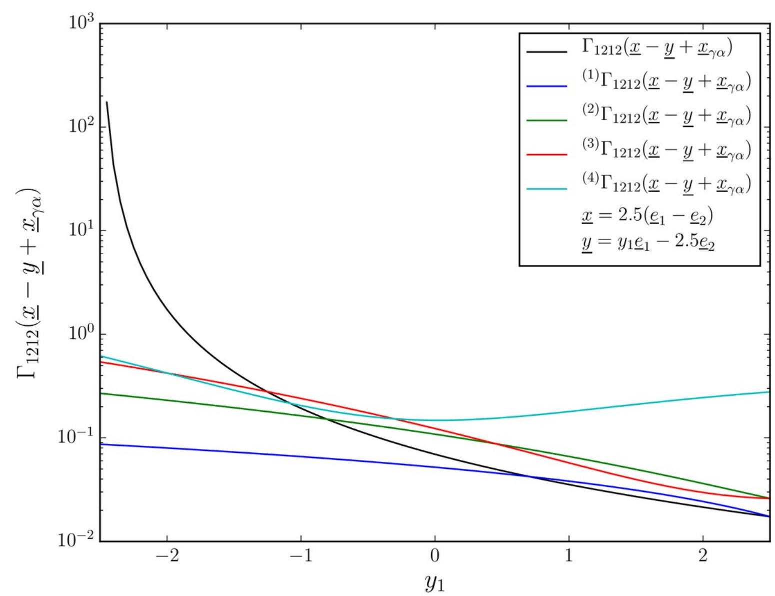
- Problems identified:
  - The Taylor expansion ${}^{(n)}\Gamma_{ijkl}(\underline{x}_{\gamma\alpha} + \underline{x} - \underline{y})$ of the Green operator $\Gamma_{ijkl}(\underline{x}_{\gamma\alpha} + \underline{x} - \underline{y})$ is very inaccurate for $(\underline{x}, \underline{y})$ away from $(\underline{x}_\alpha, \underline{x}_\gamma)$.

Example: Let $\Omega_\alpha := (0,5)^2$ and $\Omega_\gamma := (5,10) \times (0,5)$ with $\underline{x}_\alpha := 2.5(\underline{e}_1 + \underline{e}_2)$and $\underline{x}_\gamma := 2.5(2\underline{e}_1 + \underline{e}_2)$ so that $\underline{x}_{\gamma\alpha} = -2.5\underline{e}_1$. Then we have



Generalized moduli of the reference stiffness





Fixing the method

- Problems identified:
  - So far, we were only considering $\Gamma_{ijkl}(\Delta\underline{x})$ for $\|\Delta\underline{x}\| > 0$. Following the formalism of Torquato (1997), this is equivalent to say that we were only considering $H_{ijkl}(\Delta\underline{x})$ in

    $$\Gamma_{ijkl}(\Delta\underline{x}) = -A_{ijkl}\delta(\|\Delta\underline{x}\|) + H_{ijkl}(\Delta\underline{x})$$

    where $\int_{\mathcal{V}} H_{ijkl}(\underline{x}-\underline{x}')\mathrm{d}V_{\underline{x}'} = 0$ for star-convex $\mathcal{V} \subset \mathbb{R}^2$.

    Then, we have $\int_{\mathcal{V}} \Gamma_{ijkl}(\underline{x}-\underline{x}')\mathrm{d}V_{\underline{x}'} = A_{ijkl}$ if $\underline{x} \in \mathcal{V}$ and 0 otherwise.

    In summary, we were computing integrals of $\Gamma_{ijkl}(\Delta\underline{x})$ with an inaccurate estimate of $H_{ijkl}(\Delta\underline{x})$ while

    1) Neglecting the non-vanishing contribution of A_{ijkl}.
    2) Ignoring that some integral expressions of $H_{ijkl}(\Delta\underline{x})$ are zero.

- Solving the problem:
  - From Torquato (1997), we have $\tilde{A}_{ijkl} = \lim_{r\to 0}\int_{\theta=0}^{2\pi} \frac{1}{2}\left[G^1_{ik,j}(r,\theta) + G^1_{jk,i}(r,\theta)\right] n_l(\theta) r \mathrm{d}\theta$

    where $G^1_{ij,k}(r,\theta) = -\frac{r^{-1}}{2\pi} g^1_{ijkl}(\theta)$ so that $\tilde{A}_{ijkl} = -\frac{1}{4\pi}\int_0^{2\pi}\left[g^1_{ikj}(\theta) + g^1_{jki}(\theta)\right] n_l(\theta)\mathrm{d}\theta$.

    To enforce minor symmetry, we have $2A_{ijkl} := (\tilde{A}_{ijkl} + \tilde{A}_{ijlk})$
    (To enforce major symmetry, we have $2A^*_{ijkl} := (A_{ijkl} + A_{klij})$)

Q: Should we major symmetrize A?

Fixing the method

- … solving the problem. Let’s get back to our integral expressions for the influence tensors.
  - First, we have

$$\mathbb{T}_{0,0}^{\alpha\gamma} := \frac{1}{|\Omega|}\int_{\mathbb{R}^2} \chi_\gamma(\underline{y}) \left[\int_{\mathbb{R}^2} \chi_\alpha(\underline{x}) \mathbf{\Gamma}(\underline{x}-\underline{y}) \mathrm{d}\nu_{\underline{x}}\right] \mathrm{d}\nu_{\underline{y}}$$

$$= \frac{1}{|\Omega|}\int_{\Omega_\gamma} \left[\int_{\Omega_\alpha} \mathbf{\Gamma}(\underline{x}-\underline{y}) \mathrm{d}\nu_{\underline{x}}\right] \mathrm{d}\nu_{\underline{y}} = \frac{1}{|\Omega|}\int_{\Omega_\gamma} \left[\int_{\Omega_\alpha} -\mathbb{A}\delta(\underline{x}-\underline{y}) \mathrm{d}\nu_{\underline{x}}\right] \mathrm{d}\nu_{\underline{y}}$$

where $\int_{\Omega_\alpha} -\mathbb{A}\delta(\underline{x}-\underline{y})\mathrm{d}\nu_{\underline{x}} = \begin{cases} -\mathbb{A} & \text{if } \underline{y} \in \Omega_\alpha \\ 0 & \text{otherwise} \end{cases}$

so that $\mathbb{T}_{0,0}^{\alpha\gamma} = -\frac{\mathbb{A}}{|\Omega|}\int_{\Omega_\gamma} \chi_\alpha(\underline{y})\mathrm{d}\nu_{\underline{y}}$. Also, we have $2\chi_\alpha(\underline{y}) = \partial_k\left[\chi_\alpha(\underline{y})y_k\right] - \delta_\alpha(\underline{y})n_k(\underline{y})y_k$ which implies

$$\mathbb{T}_{0,0}^{\alpha\gamma} = \frac{\mathbb{A}}{2|\Omega|}\int_{\Omega_\gamma} \delta_\alpha(\underline{y})n_k(\underline{y})y_k \mathrm{d}\nu_{\underline{y}} - \frac{\mathbb{A}}{2|\Omega|}\int_{\Omega_\gamma} \partial_k[\chi_\alpha(\underline{y})y_k]\mathrm{d}\nu_{\underline{y}}$$

$$\mathbb{T}_{0,0}^{\alpha\gamma} = \frac{\mathbb{A}}{2|\Omega|}\int_{\Omega_\gamma} \delta_\alpha(\underline{y})n_k(\underline{y})y_k \mathrm{d}\nu_{\underline{y}} - \frac{\mathbb{A}}{2|\Omega|}\oint_{\partial\Omega_\gamma} \chi_\alpha(\underline{y})y_k n_k(\underline{y})\mathrm{d}s_{\underline{y}}$$

$$\mathbb{T}_{0,0}^{\alpha\gamma} = \frac{\mathbb{A}}{2|\Omega|}\int_{\Omega_\gamma} \delta_\alpha(\underline{y})n_k(\underline{y})y_k \mathrm{d}\nu_{\underline{y}} - \frac{\mathbb{A}}{2|\Omega|}\oint_{\partial\Omega_{\gamma\alpha}} y_k n_k(\underline{y})\mathrm{d}s_{\underline{y}}$$

$$\mathbb{T}_{0,0}^{\alpha\gamma} := -\frac{1}{|\Omega|}\int_{\partial\Omega_{\gamma\alpha}} \mathbb{A}\,\mathrm{d}s$$

$[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$

Influence tensors

- We want to compute $\overline{\boldsymbol{\tau}(\underline{x}) : [\boldsymbol{\Gamma} * \boldsymbol{\tau}](\underline{x})}$ in which the convolution

$$\boldsymbol{\Gamma} * \boldsymbol{\tau}(\underline{x}) = \int_{\mathbb{R}^2} \boldsymbol{\Gamma}(\underline{x} - \underline{x}') : \boldsymbol{\tau}(\underline{x}')\mathrm{d}\underline{x}'$$

is expressed as follows to handle the singularity of the Green operator for strains:

$$\boldsymbol{\Gamma} * \boldsymbol{\tau}(\underline{x}) = \mathbb{P} : \boldsymbol{\tau}(\underline{x}) + \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(\underline{x})} \mathbb{H}(\underline{x} - \underline{x}') : \boldsymbol{\tau}(\underline{x}')\mathrm{d}\underline{x}'$$

where $\mathbb{P}$ is the Hill polarization tensor of a ball embedded in a medium with reference stiffness $\mathbb{L}_0$, and $\mathbb{H}$ is the regular part of the Green operator for strains.

- Note that we have $\lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon(\underline{x})} \mathbb{H}(\underline{x} - \underline{x}')\mathrm{d}\underline{x}' = 0$ for all $\Omega \subset \mathbb{R}^2$ radial at $\underline{x}$.
- Case of piecewise constant trial fields, i.e. $\boldsymbol{\tau}(\underline{x}) = \sum_\alpha \chi_\alpha(\underline{x})\boldsymbol{\tau}^\alpha$:

$$\boldsymbol{\Gamma} * \boldsymbol{\tau}(\underline{x}) = \sum_\alpha \chi_\alpha(\underline{x})\mathbb{P} : \boldsymbol{\tau}^\alpha \implies \overline{\boldsymbol{\tau}(\underline{x}) : [\boldsymbol{\Gamma} * \boldsymbol{\tau}](\underline{x})} = \sum_\alpha c_\alpha \boldsymbol{\tau}^\alpha : \mathbb{P} : \boldsymbol{\tau}^\alpha \quad \text{where} \quad c_\alpha := \frac{|\Omega_\alpha|}{|\Omega|}$$

- Case of piecewise polynomial trial fields, i.e. $\boldsymbol{\tau}(\underline{x}) = \sum_\alpha \chi_\alpha(\underline{x}) \left(\boldsymbol{\tau}^\alpha + \sum_{k=1}^p \left\langle \boldsymbol{\tau}^\alpha \boldsymbol{\partial}^k, (\Delta^\alpha \underline{x})^{\otimes^k} \right\rangle_k \right)$:

The convolution becomes

$$\boldsymbol{\Gamma} * \boldsymbol{\tau}(\underline{x}) = \mathbb{P} : \boldsymbol{\tau}(\underline{x}) + \sum_\alpha \lim_{\varepsilon \to 0} \int_{\Omega_\alpha \setminus B_\varepsilon(\underline{x})} \mathbb{H}(\underline{x} - \underline{x}') : \sum_{k=1}^p \left\langle \boldsymbol{\tau}^\alpha \boldsymbol{\partial}^k, (\Delta^\alpha \underline{x}')^{\otimes^k} \right\rangle_k \mathrm{d}\underline{x}'$$

where $\Delta^\alpha \underline{x}' := \underline{x}' - \underline{x}_\alpha$.

Influence tensors

- Recall that we have $4H_{ijkl}(\underline{x}) = G^{(2)}_{ik,jl}(\underline{x}) + G^{(2)}_{il,jk}(\underline{x}) + G^{(2)}_{jk,il}(\underline{x}) + G^{(2)}_{jl,ik}(\underline{x})$

  where $G^{(2)}_{ij,kl}(\underline{x}) = \frac{r^{-2}}{2\pi} h^2_{ijkl}(\theta)$ with $\underline{x} = r(\underline{e}_1 \cos\theta + \underline{e}_2 \sin\theta)$ and $r := \|\underline{x}\|$.
  $\underline{x} = r\underline{n}$

  Let $h^2_{(ij)(kl)}(\underline{x}) := \frac{1}{4}[h^2_{ik,jl}(\underline{x}) + h^2_{il,jk}(\underline{x}) + h^2_{jk,il}(\underline{x}) + h^2_{jl,ik}(\underline{x})]$ so that $H_{ijkl}(\underline{x}) = \frac{r^{-2}}{8\pi} h^2_{(ij)(kl)}(\theta)$
  $= \frac{\|\underline{x}\|^{-2}}{8\pi} h^2_{(ij)(kl)}(\underline{n})$

- We are particularly in the following summand of the convolution:

$$ {}^k X^\alpha(\underline{x}) := \lim_{\varepsilon\to 0} \int_{\Omega_\alpha \setminus B_\varepsilon(\underline{x})} \mathbb{H}(\underline{x} - \underline{x}') : \left\langle \boldsymbol{\tau}^\alpha \boldsymbol{\partial}^k, (\Delta^\alpha \underline{x}')^{\otimes^k} \right\rangle_k \mathrm{d}\underline{x}' $$

  with components

$$ {}^k X^\alpha_{ij}(\underline{x}) = \lim_{\varepsilon\to 0} \int_{\Omega_\alpha \setminus B_\varepsilon(\underline{x})} \frac{\|\underline{x} - \underline{x}'\|^{-2}}{8\pi} h^2_{(ij)(kl)}(\underline{n}, \underline{n}') \tau^\alpha_{kl} \partial^k_{k_1 k_2 \dots k_k} \Delta^\alpha x'_{k_1} \Delta^\alpha x'_{k_2} \dots \Delta^\alpha x'_{k_k} \mathrm{d}\underline{x}' $$

  - Let's use a first change of variable $\underline{x}' = \underline{x}_\alpha + r'\underline{n}'$ such that Ω_α is radial at $\underline{x}_\alpha$. Then we have

$$ {}^k X^\alpha_{ij}(\underline{x}) = \lim_{\varepsilon\to 0} \int_{r'=\varepsilon}^{\xi_\alpha(\theta')} \int_{\theta'=0}^{2\pi} \frac{\|\underline{x} - r'\underline{n}'\|^{-2}}{8\pi} h^2_{(ij)(kl)}(\underline{n}, \underline{n}') \tau^\alpha_{kl} \partial^k_{(n_1, k-n_1)} (r')^{k+1} \cos^{n_1}(\theta') \sin^{k-n_1}(\theta') \mathrm{d}\theta' \mathrm{d}r' $$

    where Ω_α is assumed to have a boundary traced by the curve $\underline{x}' : [0, 2\pi) \to \partial\Omega_\alpha$
    $: \theta' \mapsto \xi_\alpha(\theta')\underline{n}'$

Fix the method! (1)

- Is the Taylor series expansion given by

$$^{n}\mathbf{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha}) := \mathbf{\Gamma}(\underline{x}_{\gamma\alpha}) + \sum_{k=1}^{n}\sum_{i=0}^{k}\frac{(-1)^i}{(k-i)!i!}\left\langle \mathbf{\Gamma}^{(k)}(\underline{x}_{\gamma\alpha}),(\underline{x}-\underline{x}_{\alpha})^{\otimes^{k-i}}\otimes(\underline{y}-\underline{x}_{\gamma})^{\otimes^{i}}\right\rangle_k$$

  a good estimate of $\mathbf{\Gamma}(\underline{x}-\underline{y}+\underline{x}_{\gamma\alpha})$ for $(\underline{x},\underline{y})\in\Omega_\alpha\times\Omega_\gamma$.

- Let $\Omega_\alpha := (0,5)^2$ and $\Omega_\gamma := (5,10)\times(0,5)$ with $\underline{x}_\alpha := 2.5(\underline{e}_1+\underline{e}_2)$ and $\underline{x}_\gamma := 2.5(2\underline{e}_1+\underline{e}_2)$ so that $\underline{x}_{\gamma\alpha} = -2.5\underline{e}_1$.

- Similarly as before, we assume an anisotropic stiffness with normalized generalized moduli given by

- Then, we have

Fix the method! (2)

- A property of the convolution operator is that, when applied to the polarization field, it returns a disturbance strain with vanishing field average, namely $\overline{(\boldsymbol{\Gamma} * \boldsymbol{\tau})} = \mathbf{0}$. Similarly, for piecewise polynomial trial, we expect to have

$$\overline{(\boldsymbol{\Gamma} * \boldsymbol{\tau}^{h_p})} = \mathbf{0}$$

which can be recast in

$$\sum_{\alpha}\sum_{\gamma}\underbrace{\int_{\Omega_\alpha}\int_{\Omega_\gamma}\Gamma_{ijkl}(\underline{x}-\underline{y})\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}}_{(T^{\alpha\gamma}_{0,0})_{ijkl}}\tau^{\gamma}_{kl} + \underbrace{\int_{\Omega_\alpha}\int_{\Omega_\gamma}\Gamma_{ijkl}(\underline{x}-\underline{y})(y_r - x^{\gamma}_r)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}}_{(T^{\alpha\gamma}_{0,1})_{ijklr}}\partial_r\tau^{\gamma}_{kl}$$

$$+ \underbrace{\int_{\Omega_\alpha}\int_{\Omega_\gamma}\Gamma_{ijkl}(\underline{x}-\underline{y})(y_r - x^{\gamma}_r)(y_s - x^{\gamma}_s)\mathrm{d}\nu_{\underline{x}}\mathrm{d}\nu_{\underline{y}}}_{(T^{\alpha\gamma}_{0,2})_{ijklrs}}\partial^2_{rs}\tau^{\gamma}_{kl} + \cdots = 0$$

Thus, we expect the estimates of the influence tenors to be such that

$$\sum_{\alpha}\sum_{\gamma}\langle\mathbb{T}^{\alpha\gamma}_{0,1}, \boldsymbol{\partial}^r\boldsymbol{\tau}\rangle_r = \mathbf{0}$$

1D variational attempt

- First. Heterogeneous medium with stiffness $L(x)$

$$\sigma(x) = L(x)\varepsilon(x)$$

- Second. Comparison medium with homogeneous stiffness L_0

$$\sigma_0 = L_0\varepsilon_0$$

- Then, introduce a polarization field given by

$$\tau(x) := \sigma(x) - L_0\varepsilon(x)$$

and the disturbance strain given by

$$\varepsilon^d(x) := \varepsilon(x) - \varepsilon_0.$$

- We have $\overline{\sigma(x)\varepsilon^d(x)} = 0$.

$$\overline{\varepsilon(x)} = \varepsilon_0 + \overline{\varepsilon^d(x)}$$

$$\tau(x) = \Delta L(x)\varepsilon(x)$$

$$[\Delta L(x)]^{-1}\tau(x) = \varepsilon(x)$$

$$[\Delta L(x)]^{-1}\tau(x) = \varepsilon_0 + \varepsilon^d(x)$$

$$\omega(x)[\Delta L(x)]^{-1}\tau(x) = \omega(x)\varepsilon_0 + \omega(x)\varepsilon^d(x)$$

$$2\Pi(\tau, \varepsilon^d) := \varepsilon_0 L_0 \varepsilon_0 - \overline{\tau(\Delta L)^{-1}\tau} + \overline{\tau\varepsilon^d} + 2\overline{\tau}\varepsilon_0$$

$$\Pi(\tau, \varepsilon^d) = \varepsilon_0 L_0 \varepsilon_0 - \overline{\tau(\Delta L)^{-1}\tau} + \overline{\tau\varepsilon} - \overline{\tau}\varepsilon_0 + 2\overline{\tau}\varepsilon_0$$

$$\Pi(\tau, \varepsilon^d) = \varepsilon_0 L_0 \varepsilon_0 - \overline{\tau(\Delta L)^{-1}\tau} + \overline{\tau\varepsilon\tau}\varepsilon_0$$

1D variational attempt

- Look at the term

$$\tau^1(k_a) = \frac{\ell}{(2\pi a)^2} \sum_{r=0}^{n-1} \exp\left(-\frac{2\pi \mathrm{i} ar}{n}\right) \left[\left(\frac{2\pi \mathrm{i} a(r+1)}{n} + 1\right) \exp\left(-\frac{2\pi \mathrm{i} a}{n}\right) - \left(\frac{2\pi \mathrm{i} ar}{n} + 1\right)\right] \partial\tau_r$$

1D HS principle for piecewise polynomial polarization

- Look at the term $\overline{\tau\Gamma\tau} = \{\tau\}[\Gamma]\{\tau\}$

where $\{\tau\}^T = \begin{bmatrix} \tau_1 & \dots & \tau_{n_\alpha} & \partial\tau_1 & \dots & \partial\tau_{n_\alpha} & \dots & \partial^p\tau_1 & \dots & \partial^p\tau_{n_\alpha} \end{bmatrix}$

then $[\Gamma]_{(kn_\alpha+\alpha,\ell n_\alpha+\beta)} = \displaystyle\sum_{a=0}^{n-1} \sum_{m=-\infty}^{\infty} \sum_{r=r_{\underline{\alpha}}}^{r_{\overline{\alpha}}} \sum_{s=r_{\underline{\beta}}}^{r_{\overline{\beta}}} \Re\left\{ {}^k f^*_{\alpha,r}(a+mn)\ {}^\ell f_{\beta,s}(a+mn)\right\} \widehat{\Gamma}(k_{a+mn})$

in which $\quad {}^k f_{\alpha,r}(a+mn) = \dfrac{1}{L}\displaystyle\int_{\frac{rL}{n}}^{\frac{(r+1)L}{n}} (x-x_\alpha)^k \exp\left[-\frac{\mathrm{i}2\pi a x}{L}\right] dx$

$$ {}^k f_{\alpha,r}(a+mn) = \frac{1}{L}\sum_{i=0}^{k}\binom{k}{i} x_\alpha^i \left[\exp\left(-\frac{\mathrm{i}2\pi a x}{L}\right)\sum_{j=0}^{k-i}(-1)^{k-j}\frac{(k-i)!}{j!}\left(-\frac{\mathrm{i}2\pi a}{L}\right)^{i+j-k-1} x^j\right]_{\frac{rL}{n}}^{\frac{(r+1)L}{n}} $$

$$ {}^k f_{\alpha,r}(a+mn) = \frac{1}{L}\sum_{i=0}^{k}\binom{k}{i} x_\alpha^i \left[\exp\left(-\frac{\mathrm{i}2\pi a x}{L}\right)\sum_{j=0}^{k-i}(-1)^{i-1}\frac{(k-i)!}{j!}\left(\frac{\mathrm{i}2\pi a}{L}\right)^{i+j-k-1} x^j\right]_{\frac{rL}{n}}^{\frac{(r+1)L}{n}} $$

To do list & Questions

- To do:
  - Enforce local equilibrium on the system
  - Verify numerical results of D/T for array of squares // (anti-)symmetry
  - Verify prescribed average strain is recovered

Questions:
- Are the global systems ever singular?
- What is the effect of truncation of the expansion of the Green operator, i.e. *n*?
- Can we truncate the level of interaction by neglecting influence tensor components of remote inclusions?
- What about nonlinear behaviors? For $r>0$, the compliance moduli will not be uniform within inclusions? What are the consequences on the method?
- Nonlinear HS variational principle, see Talbot and Willis (1985)

General remarks:
- Brisard (2011) p. 45:
  - Are you posing the system correctly for p>=1?
    - D.6c
    - 2.6b, 2.12

- Brisard (2011) p.45:
  - *Method of equivalent inclusion* vs *method of polarized inclusions*
  - What we do is analogous to the *method of polarized inclusions*
  - Convergence guaranteed for the *method of polarized inclusions*

- Brisard et al. (2014) is key in stating convergence properties of the method using a variational formulation

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