CERFACS

EUROPEAN CENTRE FOR RESEARCH AND ADVANCED TRAINING IN SCIENTIFIC COMPUTING

Recycling Krylov subspace strategies to solve stochastic elliptic equations

Presented by

N. Venkovic

venkovic@cerfacs.fr

Cerfacs

In collaboration with

P. Mycek
mycek@cerfacs.fr
Cerfacs

L. Giraud
luc.giraud@inria.fr
Inria Bordeaux

O. Le Maître
olm@limsi.fr
Limsi-CNRS

Elliptic equations with random coefficients

- Let $a(x,\theta)$ be a random coefficient field on a domain Ω, and consider the equation

$$\nabla \cdot (a(x,\theta)\nabla u(x,\theta)) = -f(x), \quad x \in \Omega, \quad \theta \in \Theta$$
$$\mathcal{B}(x,u) = 0 \qquad , \quad x \in \partial\Omega$$

with deterministic forcing and boundary conditions.

- Moments and other quantities of interest about the solution $u(x,\theta)$ can be estimated from a sample of random solutions $\{u^{(s)}(x,\theta)\}_{s=1}^{n_s}$.

For example, the Monte Carlo (MC) estimator of $\mathbb{E}[u(x)]$ is given by

$$\overline{u}_{n_s}(x,\theta) := \sum_{s=1}^{n_s} u^{(s)}(x,\theta)/n_s$$

with a sampling error depending on n_s.

- Reducing the error of an estimator generally requires drawing realizations of a very large sample of solutions.

Sequences of linear systems

- Let's consider a 1D domain. The results presented here are for stationary lognormal coefficient fields $a(x,\theta)$ such that

$$\log a(x,\theta) \sim G(x,\theta)$$

  where *G* has 0-mean and a square-exponential covariance

$$\mathbb{E}[G(x)G(x')] = \sigma^2 \exp(-(x-x')^2/2L)$$

  with variance σ^2 and correlation length L.

- An estimator of interest is evaluated upon drawing realizations of $\{a^{(s)}(x)\}_{s=1}^{n_s}$ and solving for $\{u^{(s)}(x)\}_{s=1}^{n_s}$ such that

$$\partial_x(a(x,\theta)\partial_x u(x,\theta)) = -1 \ , \quad x \in (0,1)$$

  as we let $u^{(s)}(0) = 0$ and $\partial_x u^{(s)}(1) = 0$.

- After discretization into finite elements, a system of the form $A^{(s)}x^{(s)} = b$ is obtained for each $a^{(s)}(x)$, so that the evaluation of the estimator requires to solve a long sequence of n_s linear systems.

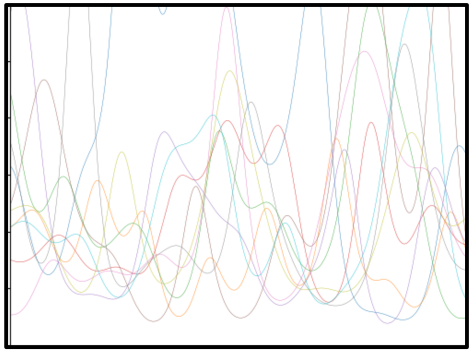
Objective

- Questions we want to answer:

  - Can we accelerate the iterative resolution of a current system $A^{(s)}x^{(s)} = b$ by recycling information from the resolution of subsequent systems $A^{(t<s)}x^{(t<s)} = b$?

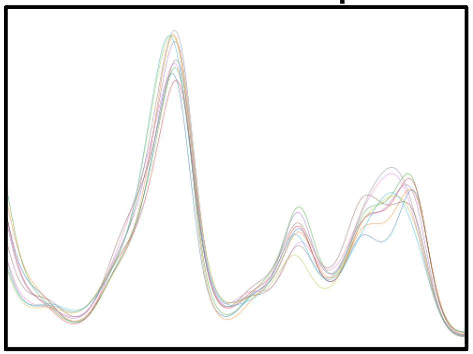
  - If so, does the sampling strategy of the random coefficient field matter?

$\{a^{(s)}(x)\}_{s=1}^{n_s}$ are i.i.d.



VS

$\{a^{(s)}(x)\}_{s=1}^{n_s}$ contains correlated subsequences



  - Could we do so efficiently for large problems? Can we speed-up this resolution even when efficient preconditioners are already used?

Deflation of linear systems with SPD matrices

- Consider the projector $H = I_n - W(W^TAW)^{-1}W^TA$ with $W \in \mathbb{R}^{n\times k}$ made of $k \ll n$ linearly independent vectors.
- A sequence $\{x_i\}_{i=0}^{j}$ of approximations of $x^* := A^{-1}b$ is obtained by

$\hat{\mu}^*$ is the direct solution of the *reduced system*

$$W^TAW\hat{\mu}^* = W^Tb$$

$$x_i := \underbrace{W\hat{\mu}^*}_{\mathcal{R}(W)} + \underbrace{Hx_i'}_{\mathcal{R}(AW)^\perp}$$

x_i' is an iterative solution of the *deflated system*

$$H^TAx' = H^Tb$$

computed by $\mathrm{CG}(H^TA, x_0')$

- Error bounded by :

$$\|x^* - x_i\|_A \leq 2\|x^* - x_0\|_A \left(\frac{\sqrt{\kappa'(H^TA)} - 1}{\sqrt{\kappa'(H^TA)} + 1}\right)^i$$

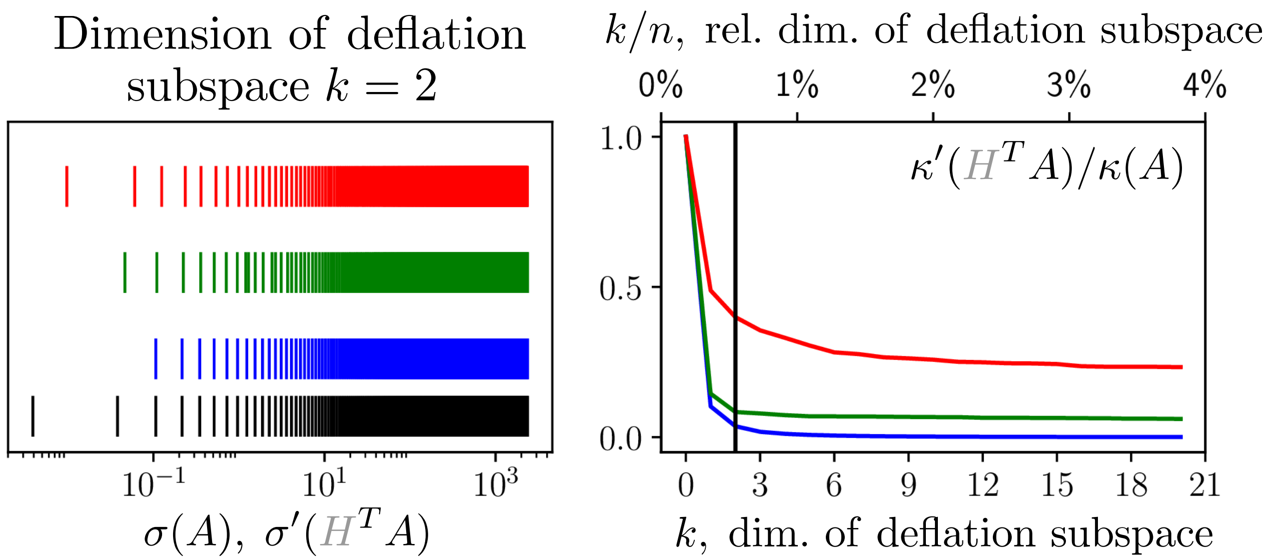
- The lower the conditioning number of the deflated system, the smaller the bound on $\|x^* - x_i\|_A$. Ideally, we would pick W to minimize $\kappa'(H^TA)$.

Saad et al. (2000), SISC 21(5):1909–1926.

What part of the spectrum should we deflate?

- A common and efficient way to deflate is to let the columns of W be eigenvectors of A.
- Let $U^{(k)} \in \mathbb{R}^{n \times k}$ contain the *k* <u>least dominant</u> eigenvectors of A . Then,

$$\color{blue}{W} = U^{(k)} \quad \color{green}{W} = U^{(k)} + \color{green}{\delta W} \quad \color{red}{W} = U^{(k)} + \color{red}{\delta W} \quad \|\color{green}{\delta W}\|_2 < \|\color{red}{\delta W}\|_2$$

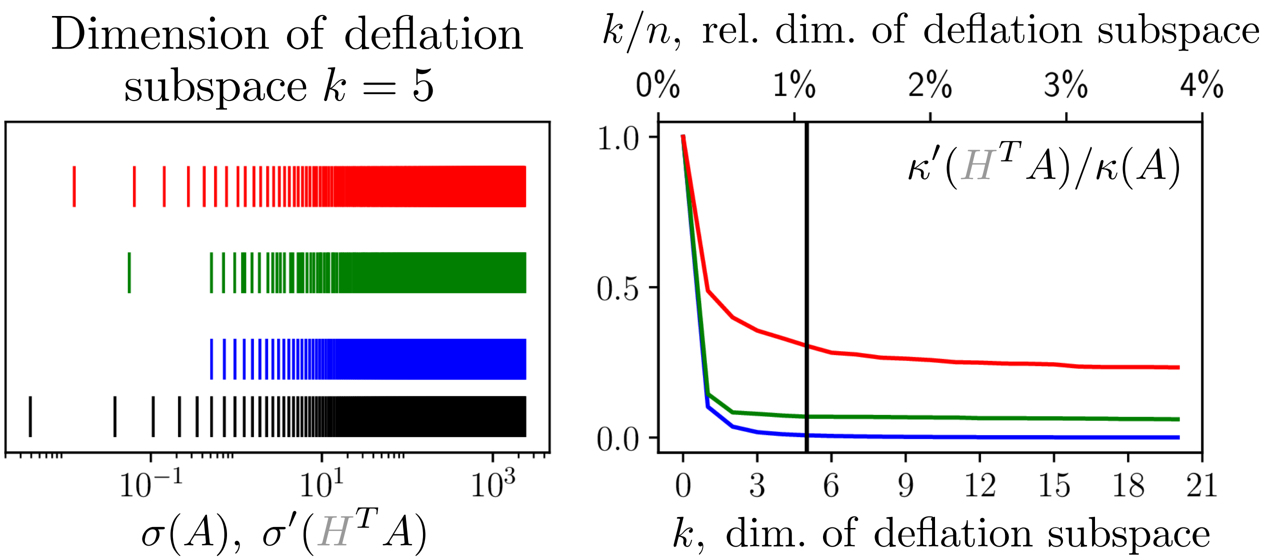


- Letting W approximate the <u>least dominant</u> eigenvectors of A significantly <u>improves the conditioning of the system</u>.

What part of the spectrum should we deflate?

- A common and efficient way to deflate is to let the columns of W be eigenvectors of A.
- Let $U^{(k)} \in \mathbb{R}^{n \times k}$ contain the *k* <u>least dominant</u> eigenvectors of A . Then,

$$W = U^{(k)} \quad W = U^{(k)} + \delta W \quad W = U^{(k)} + \delta W \quad \|\delta W\|_2 < \|\delta W\|_2$$

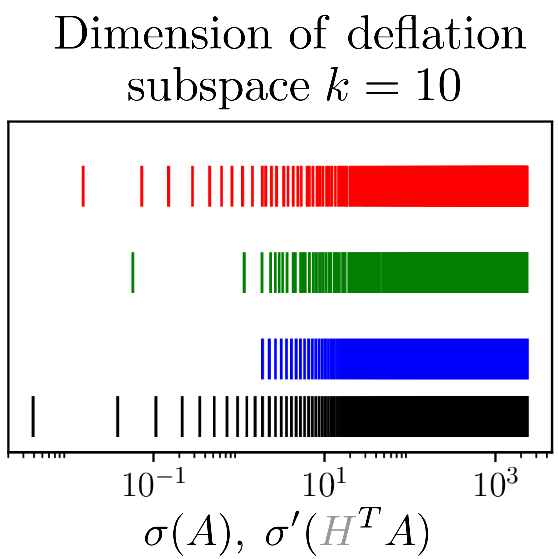


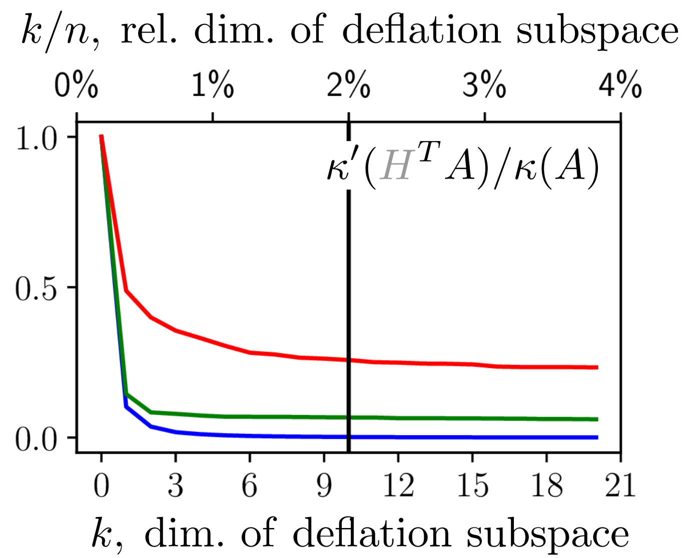
- Letting W approximate the <u>least dominant</u> eigenvectors of A significantly <u>improves the conditioning of the system</u>.

What part of the spectrum should we deflate?

- A common and efficient way to deflate is to let the columns of W be eigenvectors of A.
- Let $U^{(k)} \in \mathbb{R}^{n \times k}$ contain the *k* <u>least dominant</u> eigenvectors of A . Then,

$$W = U^{(k)} \qquad W = U^{(k)} + \delta W \qquad W = U^{(k)} + \delta W \qquad \|\delta W\|_2 < \|\delta W\|_2$$



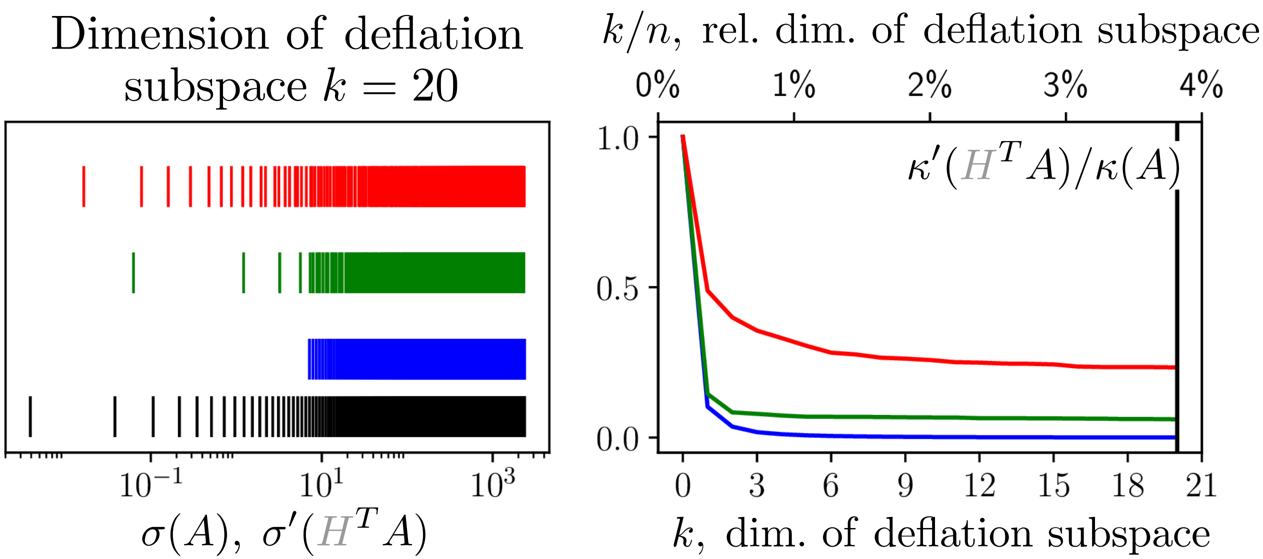


- Letting W approximate the <u>least dominant</u> eigenvectors of A significantly <u>improves the conditioning of the system</u>.

What part of the spectrum should we deflate?

- A common and efficient way to deflate is to let the columns of W be eigenvectors of A.
- Let $U^{(k)} \in \mathbb{R}^{n \times k}$ contain the *k* least dominant eigenvectors of A . Then,

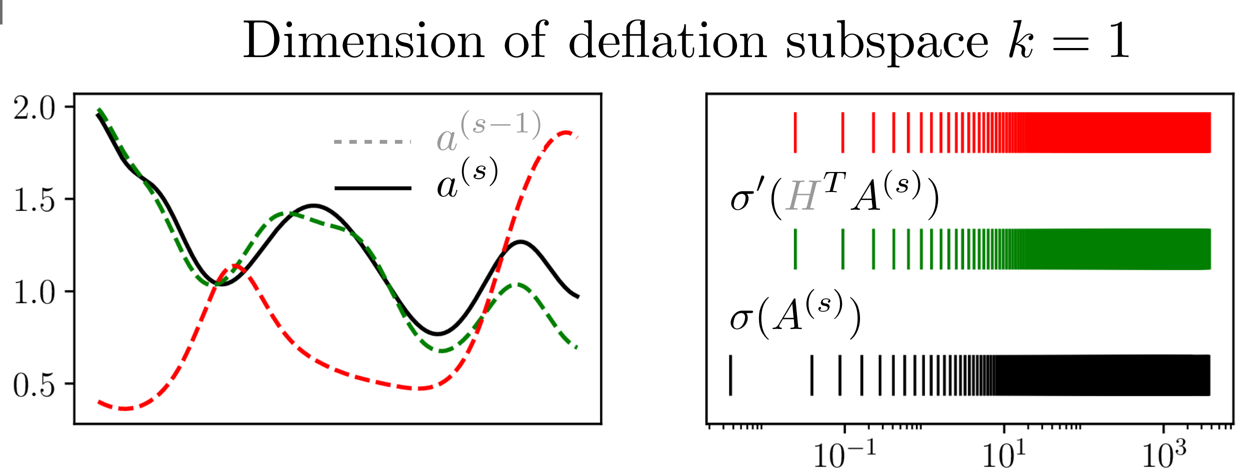
$$W = U^{(k)} \quad W = U^{(k)} + \delta W \quad W = U^{(k)} + \delta W \quad \|\delta W\|_2 < \|\delta W\|_2$$

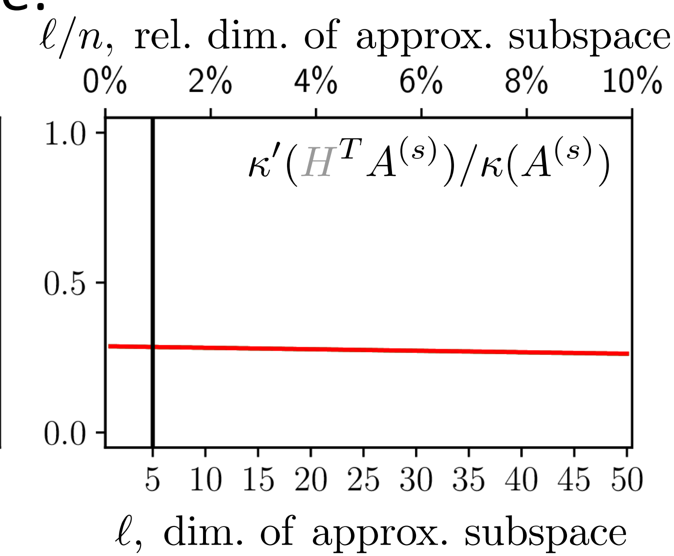


- Letting W approximate the least dominant eigenvectors of A significantly improves the conditioning of the system.

Effect of similarity between sampled fields on quality of deflation

- Deflate $A^{(s)}$ with approximate eigenvectors contained in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$. Effect of relation between $A^{(s-1)}$ and $A^{(s)}$ on the deflation process?
- Let $a^{(s)}$ be the coefficient field of $A^{(s)}$, and consider different fields $a^{(s-1)}$ of $A^{(s-1)}$. For $W^{(s-1)} = \emptyset$, we observe:

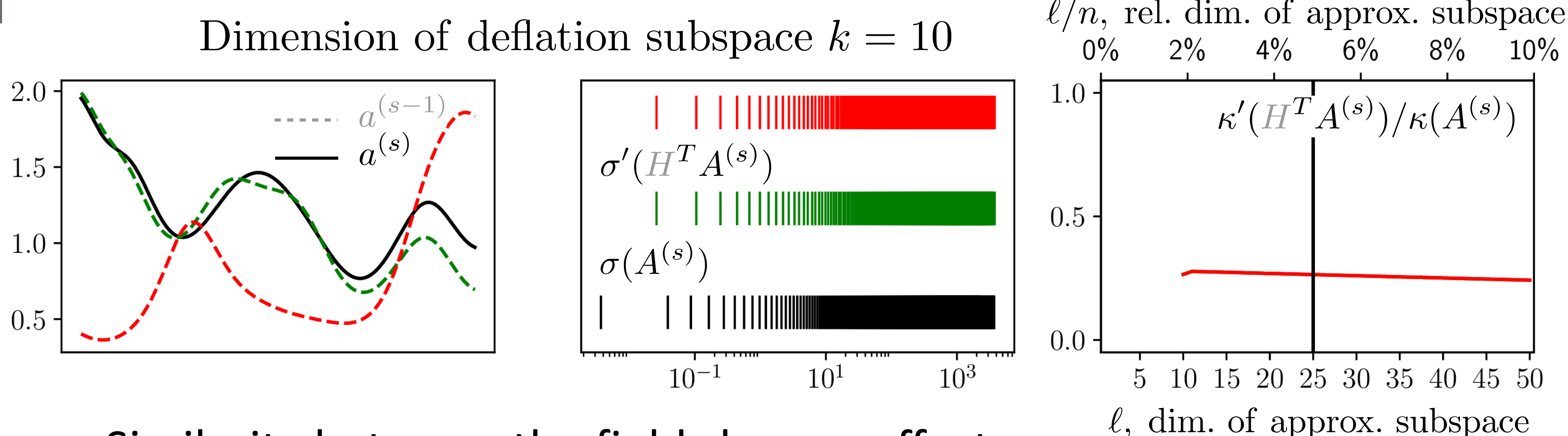




- Similarity between the fields has no effect,
- The dimension of the approximation subspace has no effect.

Effect of similarity between sampled fields on quality of deflation

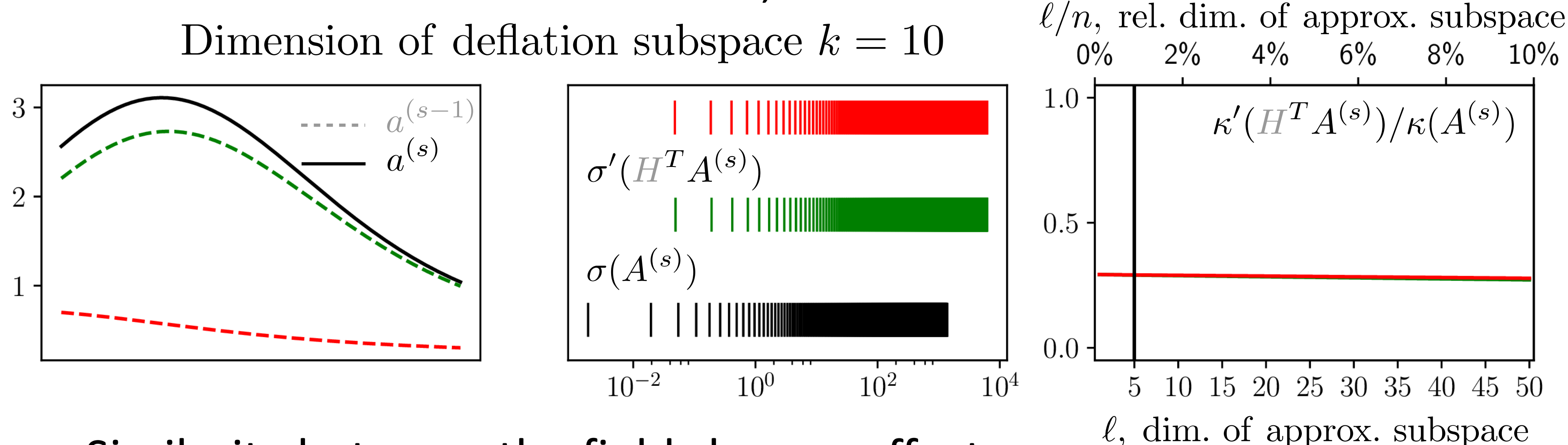
- Deflate $A^{(s)}$ with approximate eigenvectors contained in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$. Effect of relation between $A^{(s-1)}$ and $A^{(s)}$ on the deflation process?
- Let $a^{(s)}$ be the coefficient field of $A^{(s)}$, and consider different fields $a^{(s-1)}$ of $A^{(s-1)}$. For $W^{(s-1)} = \emptyset$, we observe:



- Similarity between the fields has no effect,
- The dimension of the approximation subspace has no effect.

Effect of similarity between sampled fields on quality of deflation

- Deflate $A^{(s)}$ with approximate eigenvectors contained in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$. Effect of relation between $A^{(s-1)}$ and $A^{(s)}$ on the deflation process?
- Let $a^{(s)}$ be the coefficient field of $A^{(s)}$, and consider different fields $a^{(s-1)}$ of $A^{(s-1)}$. For $W^{(s-1)} = \emptyset$, we observe:

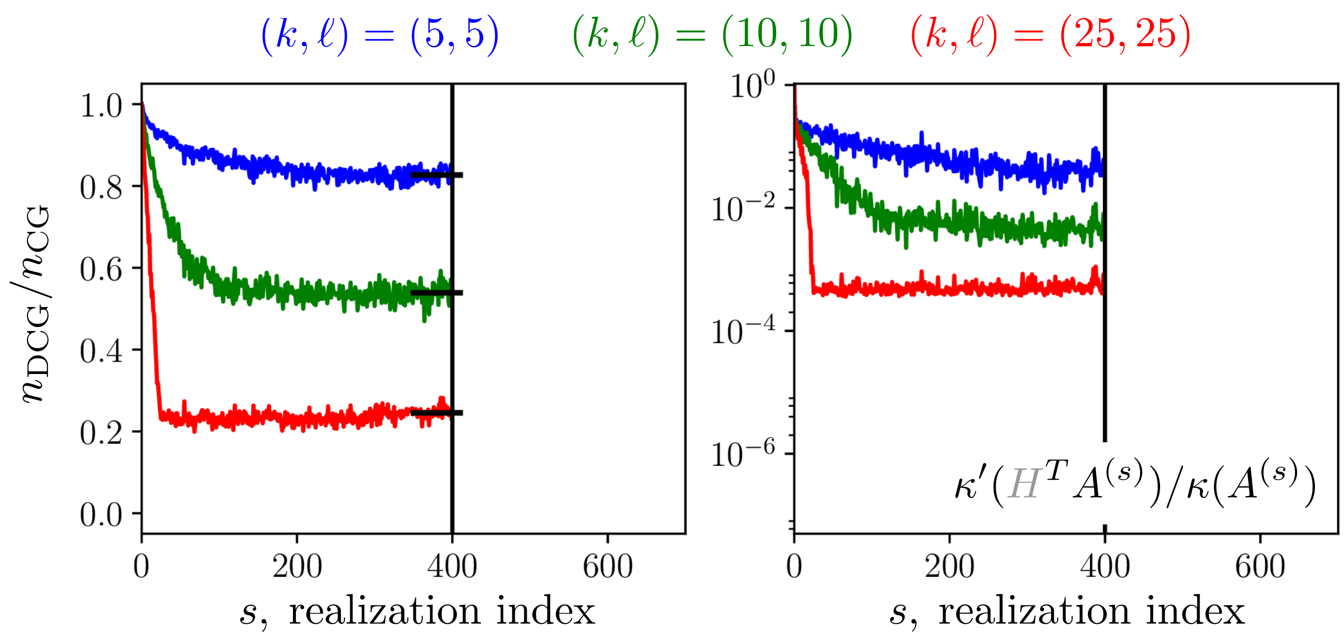


- Similarity between the fields has no effect,
- The dimension of the approximation subspace has no effect.

Solving a MC sampled sequence of systems $A^{(s)}x^{(s)} = b$ by deflated CG

- Deflate current system with approximate eigenvectors $W^{(s)} := [w_1^{(s)}, \dots, w_k^{(s)}]$ of $A^{(s)}$ built by harmonic Ritz analysis in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$.

- Considering different dimensions *k* of deflation subspaces, we observe:



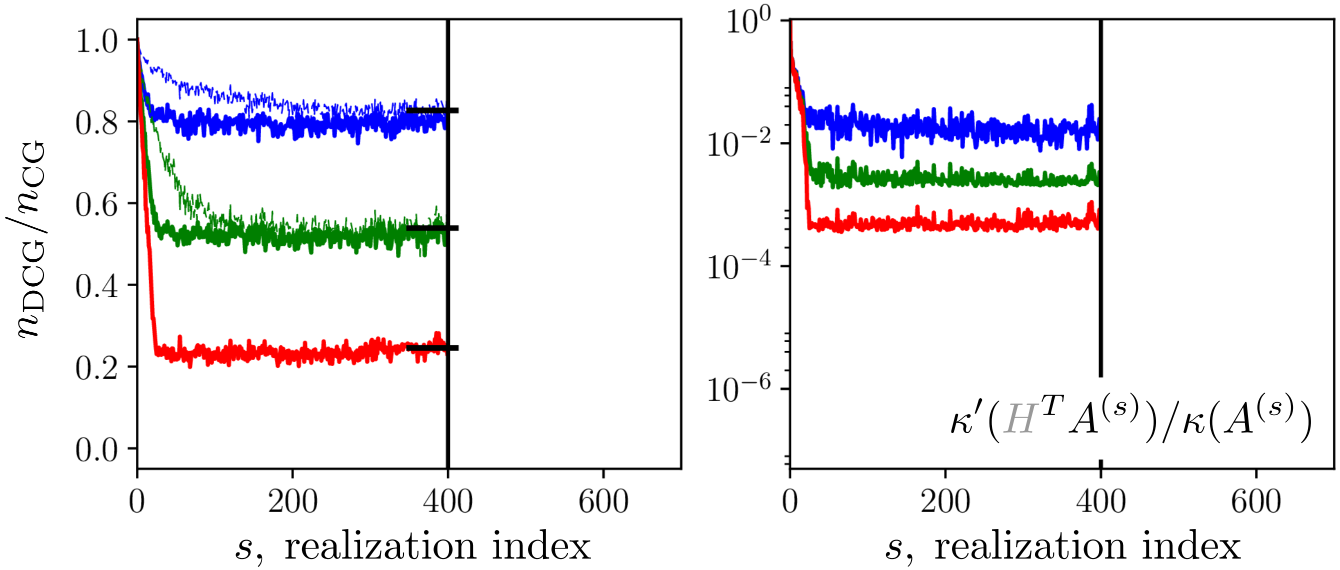
an asymptotic gain of iterations over CG which strongly depends on *k*.

Solving a MC sampled sequence of systems $A^{(s)}x^{(s)} = b$ by deflated CG

- Deflate current system with approximate eigenvectors $W^{(s)} := [w_1^{(s)}, \dots, w_k^{(s)}]$ of $A^{(s)}$ built by harmonic Ritz analysis in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$.

- Increasing the dimension ℓ of the approximation subspaces used for the harmonic Ritz anlysis, we observe:

$(k,\ell) = (5,25)$ $(k,\ell) = (10,25)$ $(k,\ell) = (25,25)$

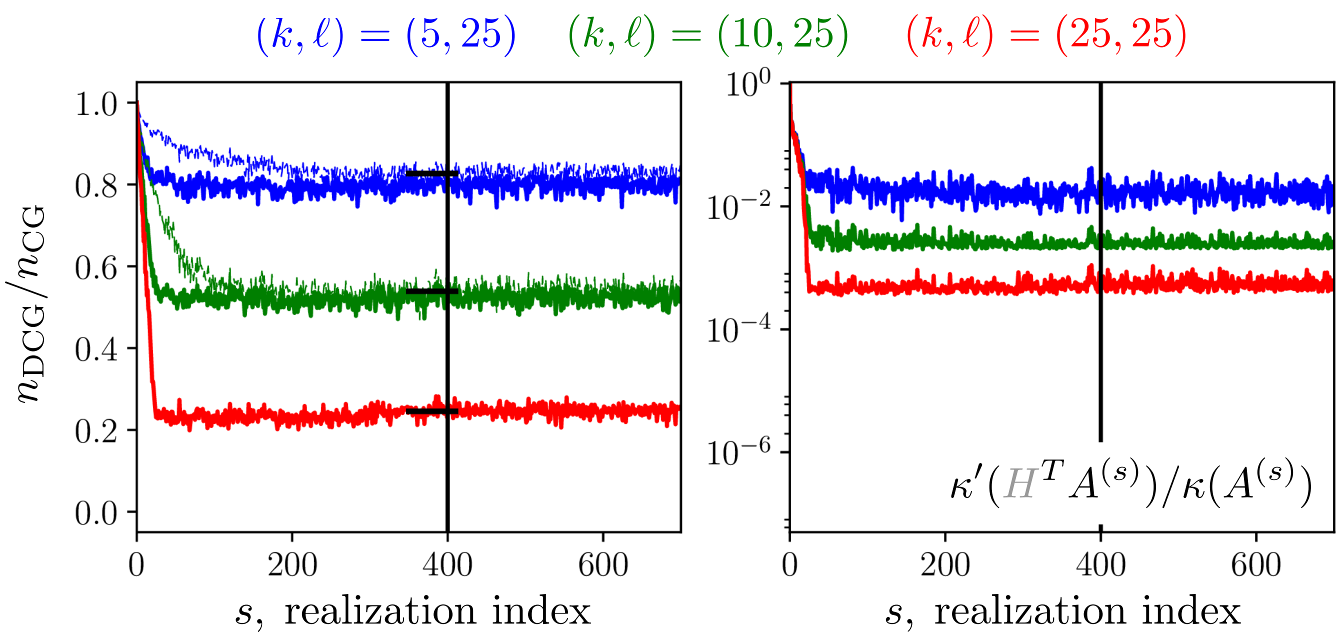


a faster gain of iterations over CG, still to the same asymptotic speed-up.

Solving a MC sampled sequence of systems $A^{(s)}x^{(s)} = b$ by deflated CG

- Deflate current system with approximate eigenvectors $W^{(s)} := [w_1^{(s)}, \ldots, w_k^{(s)}]$ of $A^{(s)}$ built by harmonic Ritz analysis in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$.

- After stopping the update of the deflation subspace, we observe

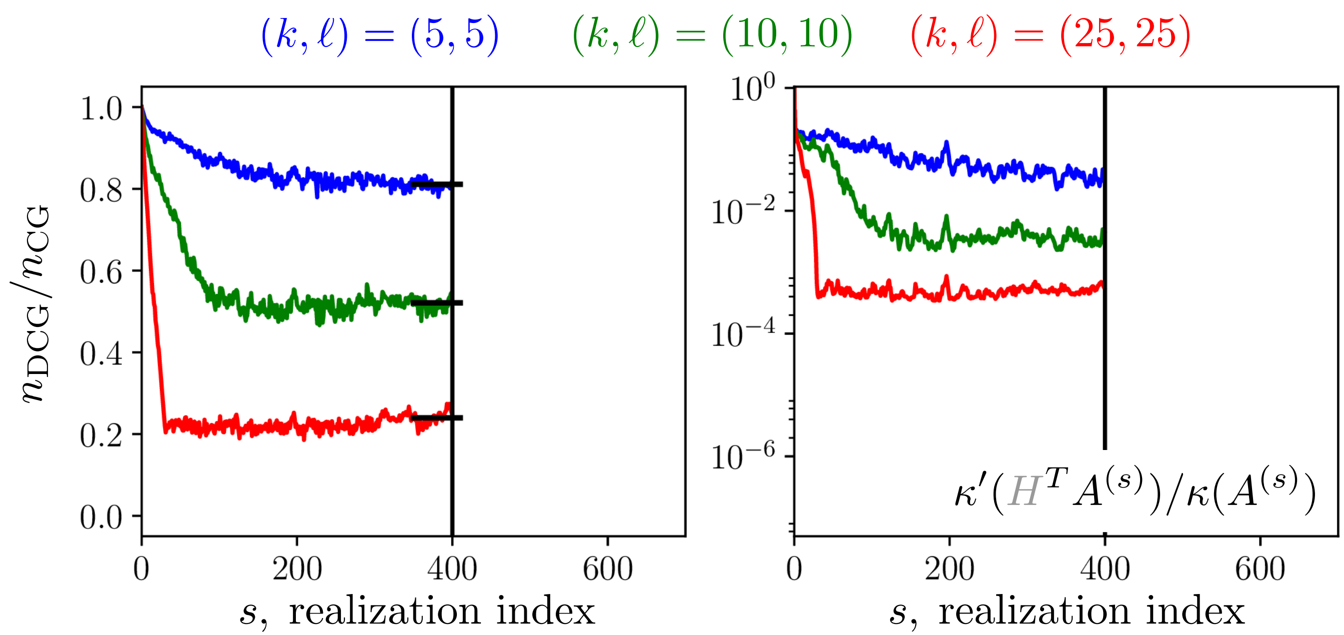


the relative gain of iteration is preserved.

Solving a MCMC sampled sequence of systems $A^{(s)}x^{(s)} = b$ by deflated CG

- Deflate current system with approximate eigenvectors $W^{(s)} := [w_1^{(s)}, \dots, w_k^{(s)}]$ of $A^{(s)}$ built by harmonic Ritz analysis in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$.

- When sampling the fields of coefficients by Markov Chain Monte Carlo,

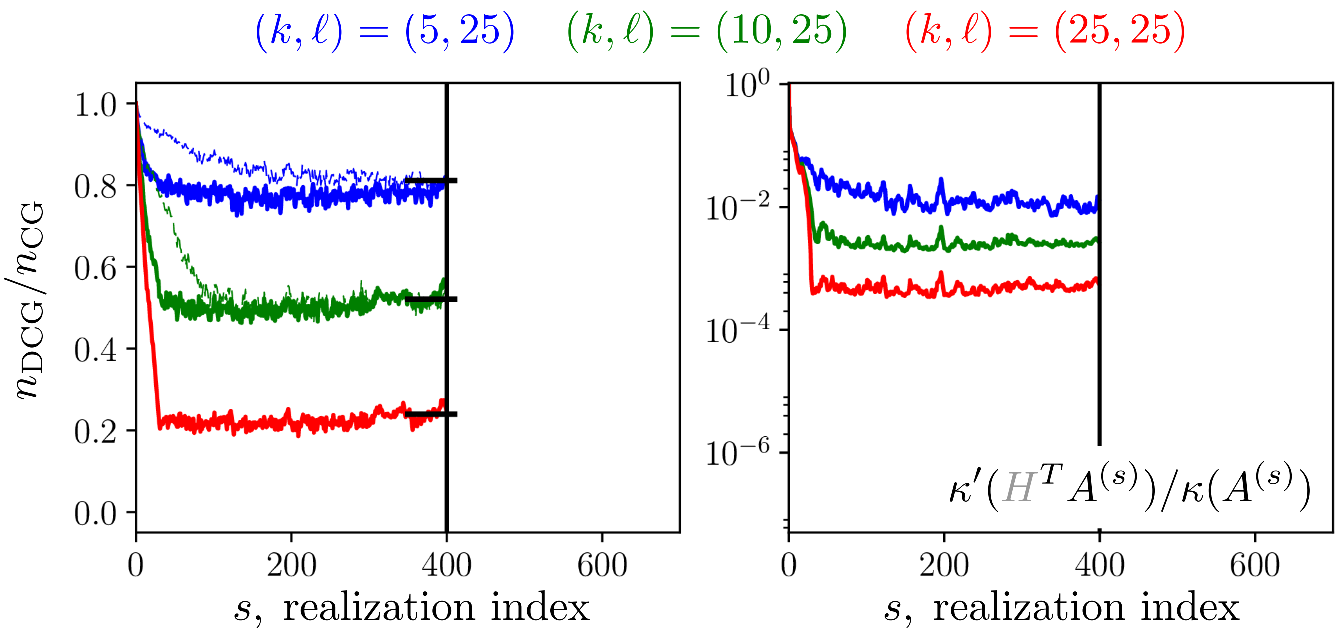


a similar behavior is observed.

Solving a MCMC sampled sequence of systems $A^{(s)}x^{(s)} = b$ by deflated CG

- Deflate current system with approximate eigenvectors $W^{(s)} := [w_1^{(s)}, \ldots, w_k^{(s)}]$ of $A^{(s)}$ built by harmonic Ritz analysis in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$.

- When sampling the fields of coefficients by Markov Chain Monte Carlo,

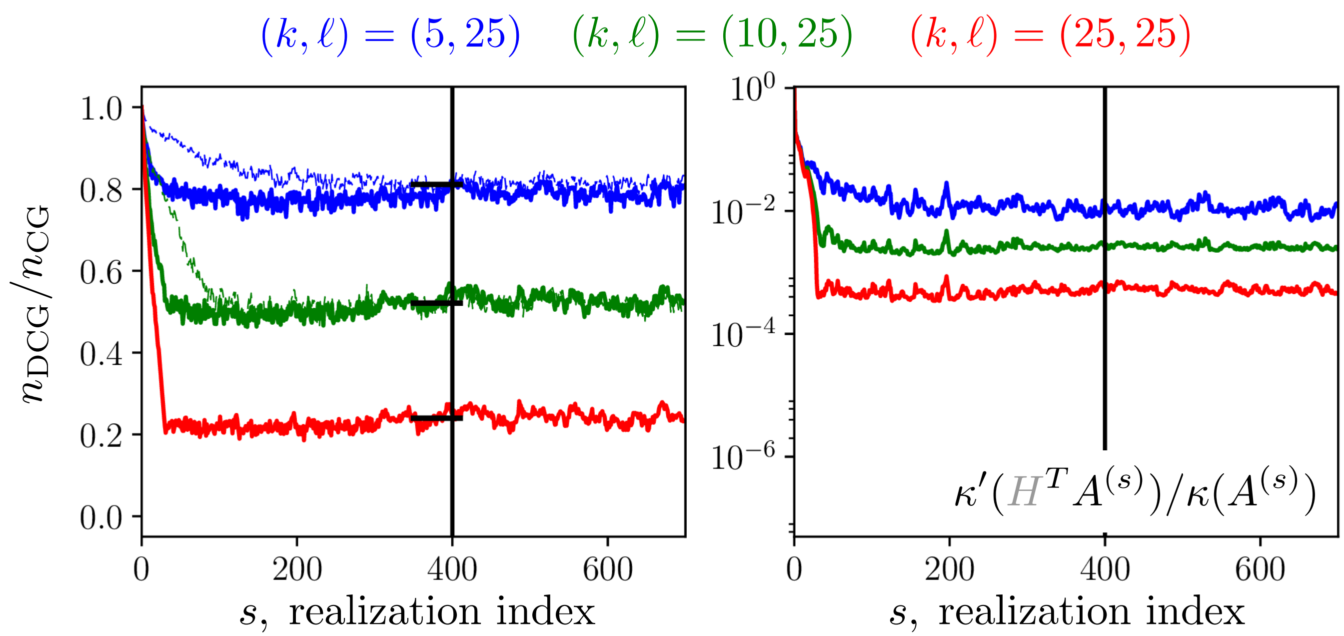


a similar behavior is observed.

Solving a MCMC sampled sequence of systems $A^{(s)}x^{(s)} = b$ by deflated CG

- Deflate current system with approximate eigenvectors $W^{(s)} := [w_1^{(s)}, \ldots, w_k^{(s)}]$ of $A^{(s)}$ built by harmonic Ritz analysis in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$.

- When sampling the fields of coefficients by Markov Chain Monte Carlo,



a similar behavior is observed.

Deflation and preconditioning of SPD systems

- Consider a preconditioner $M = LL^T$. Let $\dot{A} := L^{-1}AL^{-T}$ and define the projector $\dot{H} = I_n - \dot{W}(\dot{W}^T\dot{A}\dot{W})^{-1}\dot{W}^T\dot{A}$ with $k \ll n$ linearly independent vectors $[\dot{w}_1, \ldots, \dot{w}_k] =: \dot{W}$.
- A sequence $\{x_i\}_{i=0}^{j}$ of approximations of $x^* := A^{-1}b$ is obtained by

$\hat{\mu}^*$ is the direct solution of a *reduced system*

$$x_i := L^{-T}(\dot{W}\hat{\mu}^* + \dot{H}x_i')$$

x_i' is an iterative solution of the *deflated system*

$$\dot{H}^T\dot{A}x' = \dot{H}^T L^{-1}b$$

- Error bounded by $\|x^* - x_i\|_A \leq 2\|x^* - x_0\|_A \left(\frac{\sqrt{\kappa'(\dot{H}^T\dot{A})} - 1}{\sqrt{\kappa'(\dot{H}^T\dot{A})} + 1} \right)^i$
- Two heuristics are considered for the choice of $\dot{W}$:

(1) Deflated Preconditioned CG (DPCG)

(2) Preconditioned Deflated CG (PDCG)

Solving a sampled sequence of systems by DPCG or PDCG with exact eigenvectors

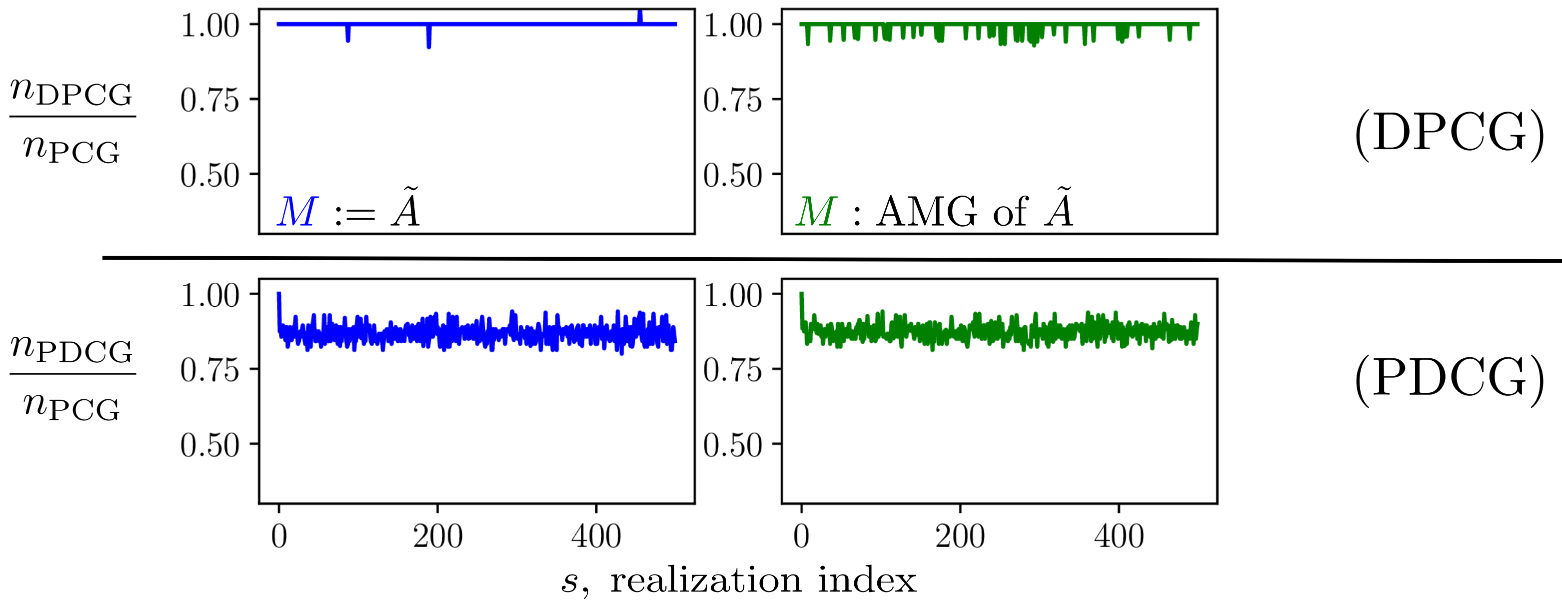
- Consider two preconditioners :

$$(1)\ M := \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 2.8\% \text{ on average}$$

$$(2)\ M : \text{AMG of } \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 3.1\% \text{ on average}$$

and compare their relative performance with DPCG and PDCG.

- Considering a relative dimension of deflation subspace $k/n = 0.4\%$, i.e. $k = 2$:



Solving a sampled sequence of systems by DPCG or PDCG with exact eigenvectors

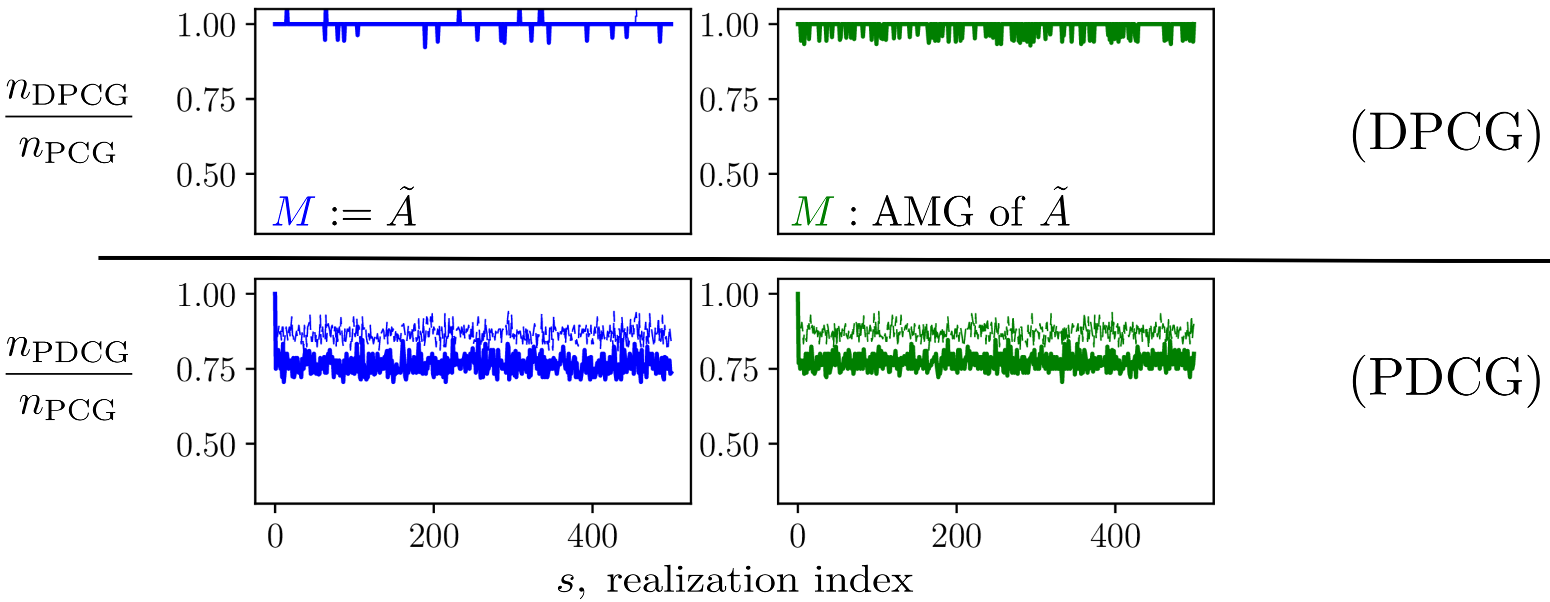
- Consider two preconditioners :

$$(1)\ M := \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 2.8\% \text{ on average}$$

$$(2)\ M : \text{AMG of } \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 3.1\% \text{ on average}$$

and compare their relative performance with DPCG and PDCG.

- Considering a relative dimension of deflation subspace $k/n = 2\%$, i.e. $k = 10$:



AMG : Algebraic multigrid

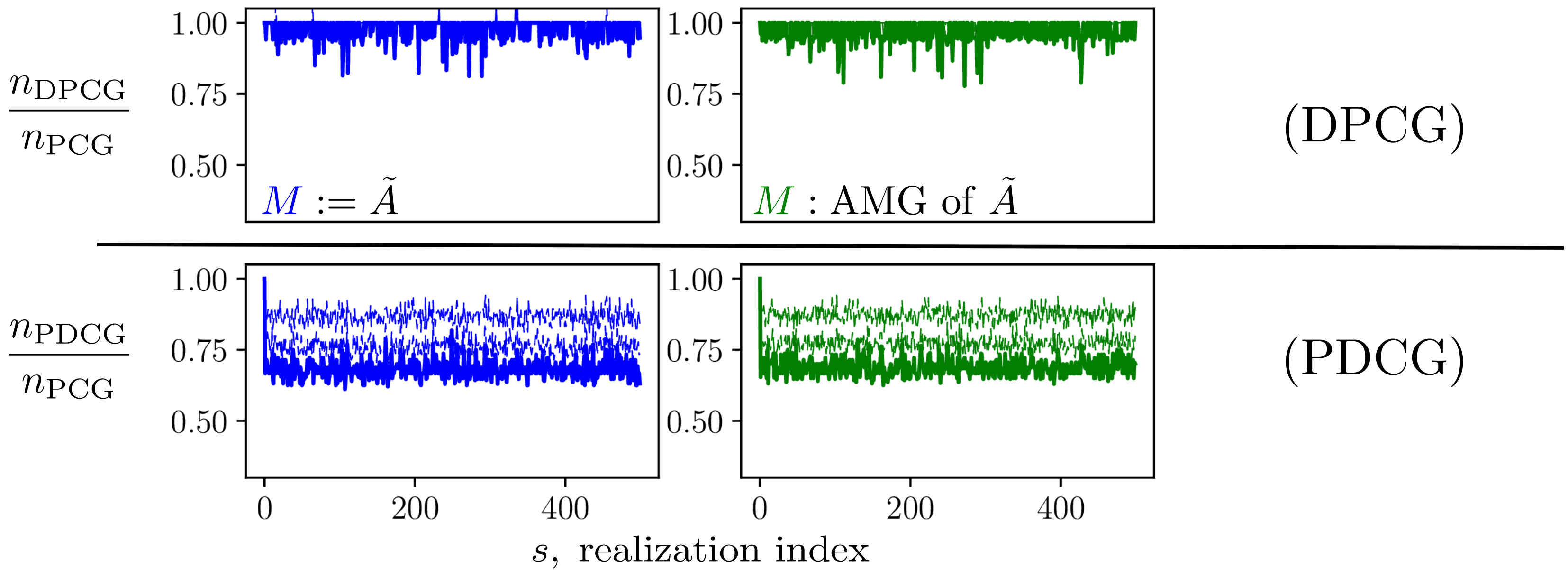
- Consider two preconditioners :

$$(1)\ M := \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 2.8\% \text{ on average}$$

$$(2)\ M : \text{AMG of } \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 3.1\% \text{ on average}$$

and compare their relative performance with DPCG and PDCG.

- Considering a relative dimension of deflation subspace $k/n = 8\%$, i.e. $k = 40$:



AMG : Algebraic multigrid

Solving a sampled sequence of systems by DPCG or PDCG with exact eigenvectors

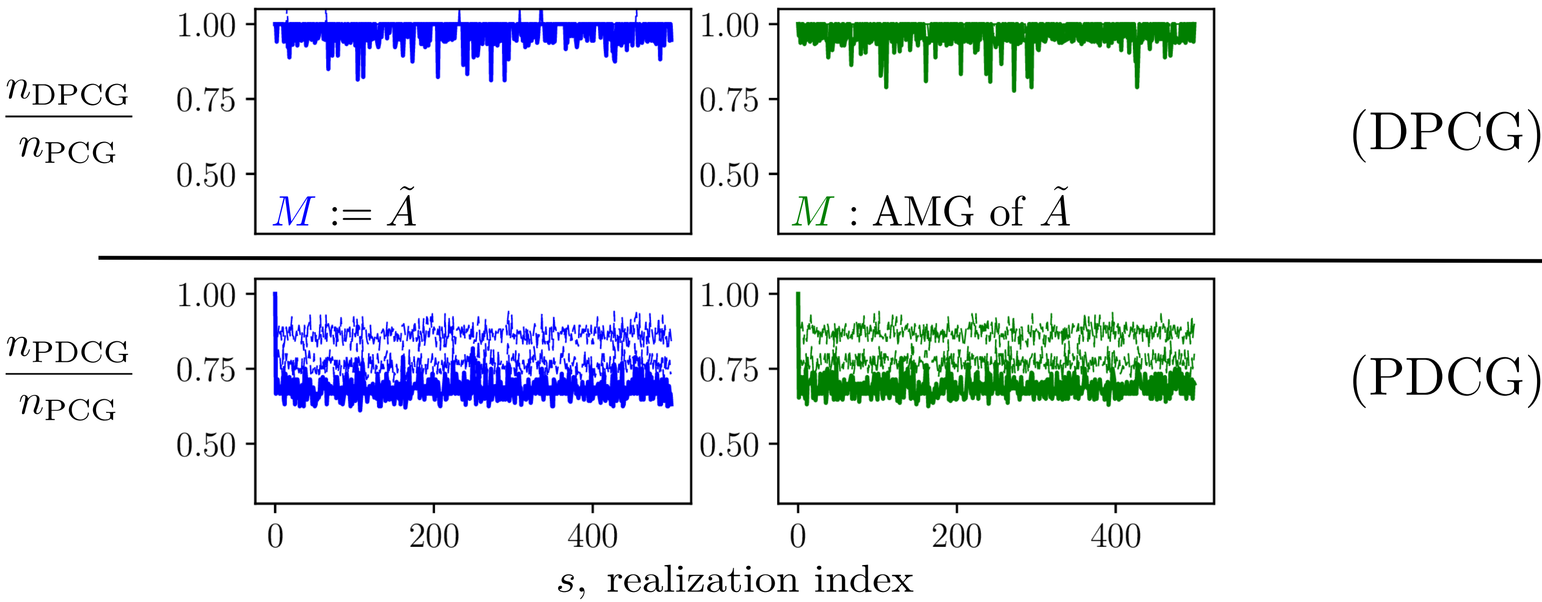
- Consider two preconditioners :

$$(1)\ M := \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 2.8\% \text{ on average}$$

$$(2)\ M : \text{AMG of } \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 3.1\% \text{ on average}$$

and compare their relative performance with DPCG and PDCG.

- Considering a relative dimension of deflation subspace $k/n = 8\%$, i.e. $k = 40$:



Deflating with the AMG preconditioner works as well as with the median operator,

Solving a sampled sequence of systems by DPCG or PDCG with exact eigenvectors

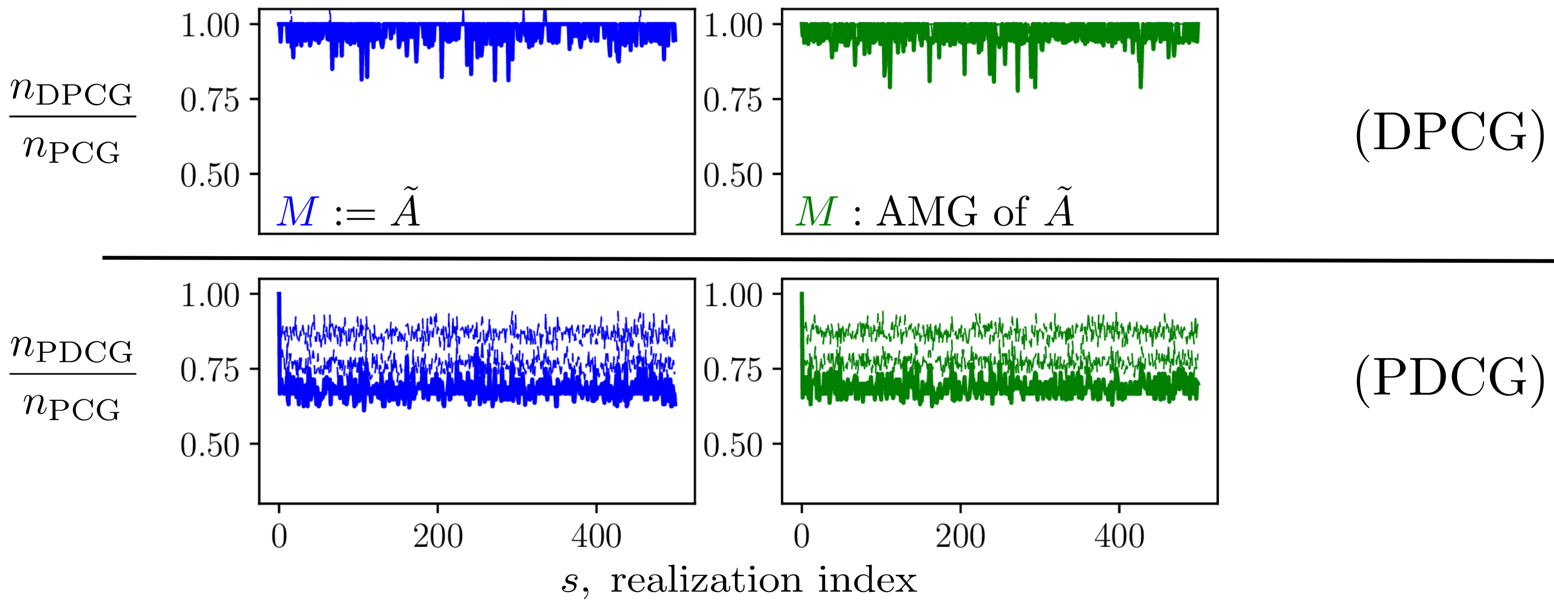
- Consider two preconditioners :

$$(1)\ M := \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 2.8\% \text{ on average}$$

$$(2)\ M : \text{AMG of } \tilde{A} \implies n_{\mathrm{PCG}}/n_{\mathrm{CG}} = 3.1\% \text{ on average}$$

and compare their relative performance with DPCG and PDCG.

- Considering a relative dimension of deflation subspace $k/n = 8\%$, i.e. $k = 40$:

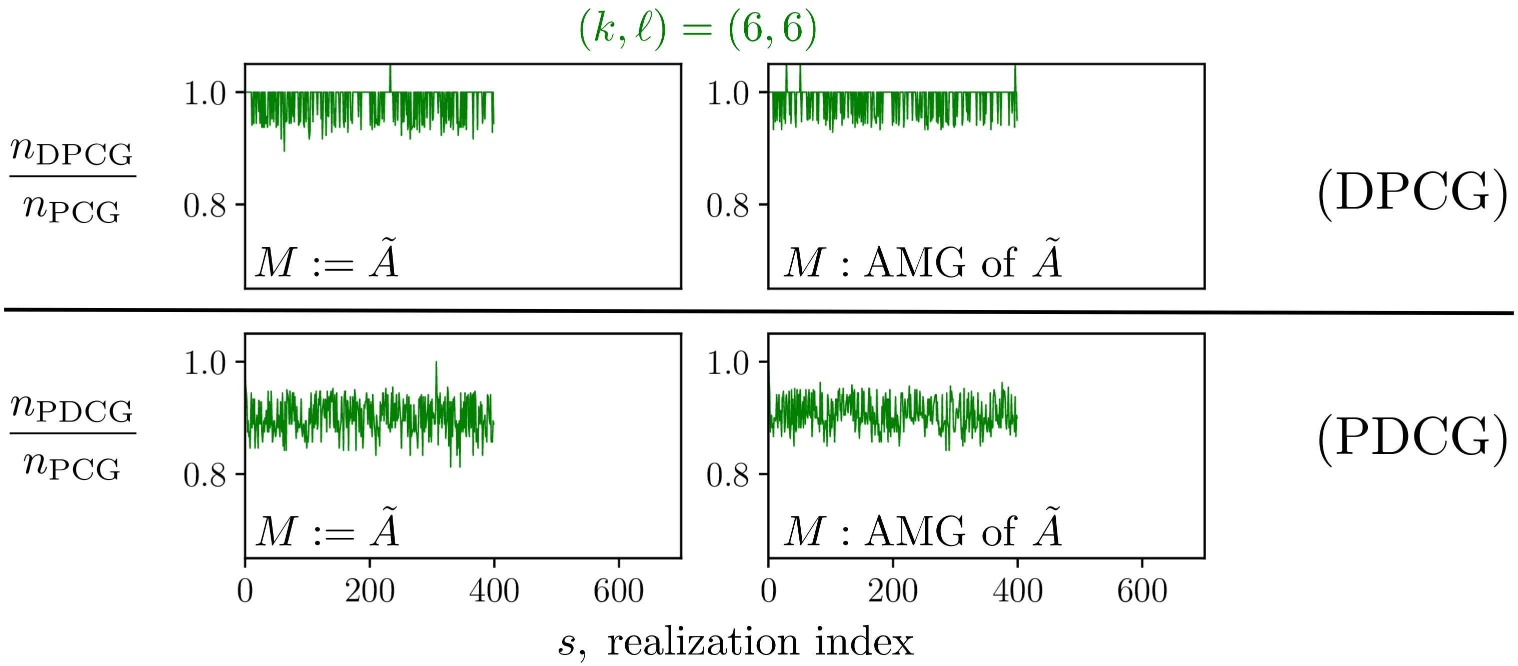


Deflating before preconditioning (PDCG) works better than the opposite (DPCG).

AMG : Algebraic multigrid

Solving a sampled sequence of systems by DPCG or PDCG with approximate eigenvectors

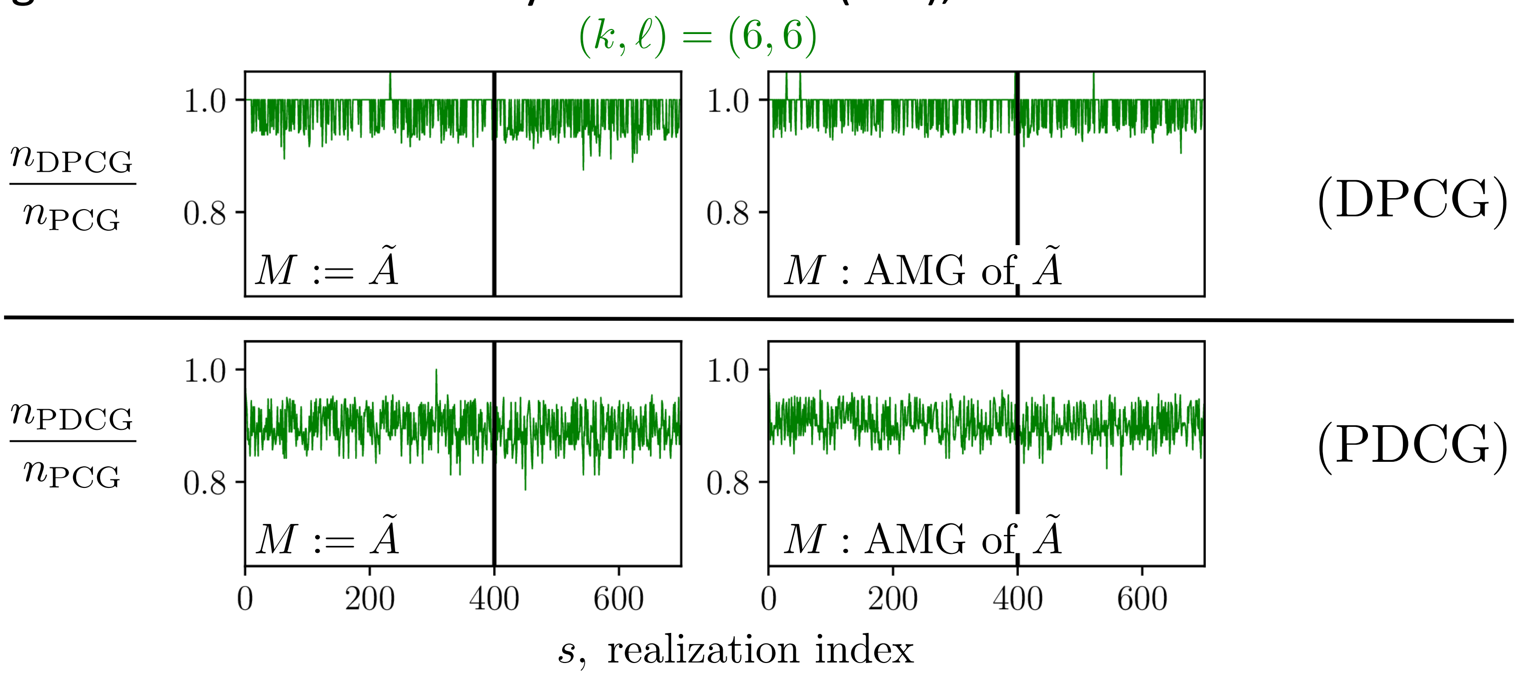
- Deflate current system with approximate eigenvectors $\dot{W}^{(s)} := [\dot{w}_1^{(s)}, \ldots, \dot{w}_k^{(s)}]$ of $L^{-1}A^{(s)}L^{-T}$ (DPCG), or $L^{T}A^{(s)}L^{-T}$ (PDCG), obtained by harmonic Ritz analysis in an approximation subspace of dimension $(k+\ell)$ obtained while solving the precedent system $A^{(s-1)}x^{(s-1)} = b$.
- Sampling the coefficient fields by Monte Carlo (MC), we obtain:



The asymptotic speed-up is quickly reached. PDCG still yields a greater acceleration than DPCG. The AMG preconditioner is as good as the median.

Solving a sampled sequence of systems by DPCG or PDCG with approximate eigenvectors

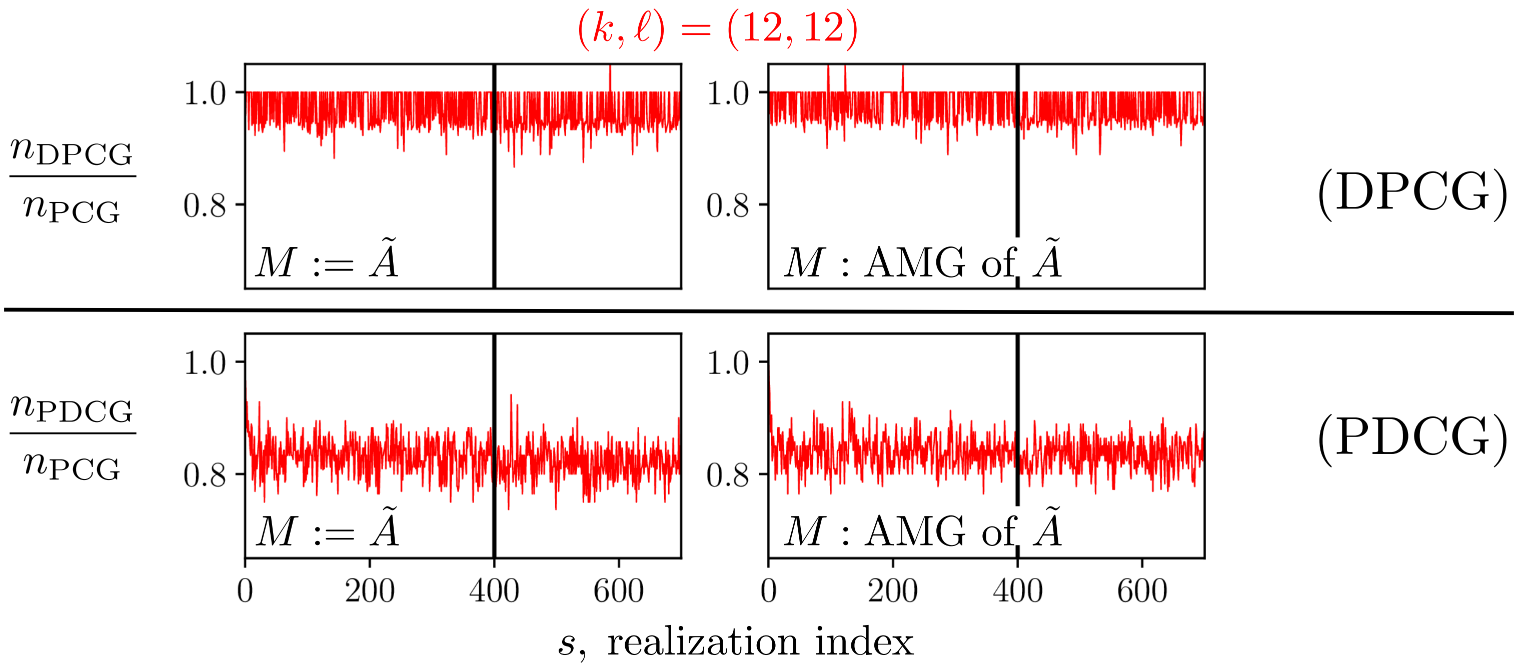
- Deflate current system with approximate eigenvectors $\dot{W}^{(s)} := [\dot{w}_1^{(s)}, \ldots, \dot{w}_k^{(s)}]$ of $L^{-1}A^{(s)}L^{-T}$ (DPCG), or $L^{T}A^{(s)}L^{-T}$ (PDCG), obtained by harmonic Ritz analysis in an approximation subspace of dimension $(k+\ell)$ obtained while solving the precedent system $A^{(s-1)}x^{(s-1)} = b$.
- Sampling the coefficient fields by Monte Carlo (MC), we obtain:



Similarly as for the non-preconditioned case, the deflation subspace need not be updated to preserve the asymptotic speed-up.

Solving a sampled sequence of systems by DPCG or PDCG with approximate eigenvectors

- Deflate current system with approximate eigenvectors $\dot{W}^{(s)} := [\dot{w}_1^{(s)}, \ldots, \dot{w}_k^{(s)}]$ of $L^{-1}A^{(s)}L^{-T}$ (DPCG), or $L^T A^{(s)} L^{-T}$ (PDCG), obtained by harmonic Ritz analysis in an approximation subspace of dimension $(k+\ell)$ obtained while solving the precedent system $A^{(s-1)}x^{(s-1)} = b$.
- Sampling the coefficient fields by Monte Carlo (MC), we obtain:



a similar behavior is observed when increasing the dimension of the deflation subspace.

Conclusions

- Deflation is a low-cost solution to accelerate the iterative resolution of sequences of linear systems which arise when evaluating estimators based on solutions of random elliptic equations
- The sampling strategy of the random field has little impact on the quality of the deflation
- Once a stable iteration gain is reached, there is no need to keep updating the deflation subspace
- Using an adequate deflation heuristic allows to reach iteration gains over PCG when paired with preconditioners based on a median operator
- Even when using a preconditioner whose parallel application is known to be more scalable than the median operator itself

Perspectives

- 2D problems
- Domain decomposition
- Parallel implementation

Thank you

Eigvenvector approximation

- Least dominant eigenvectors of $A^{(s)}$ are commonly approximated with harmonic Ritz vectors w in some subspace $\mathcal{S}^{(s)}$ s.t.

$$A^{(s)}w - \theta w \perp A^{(s)}\mathcal{S}^{(s)}$$

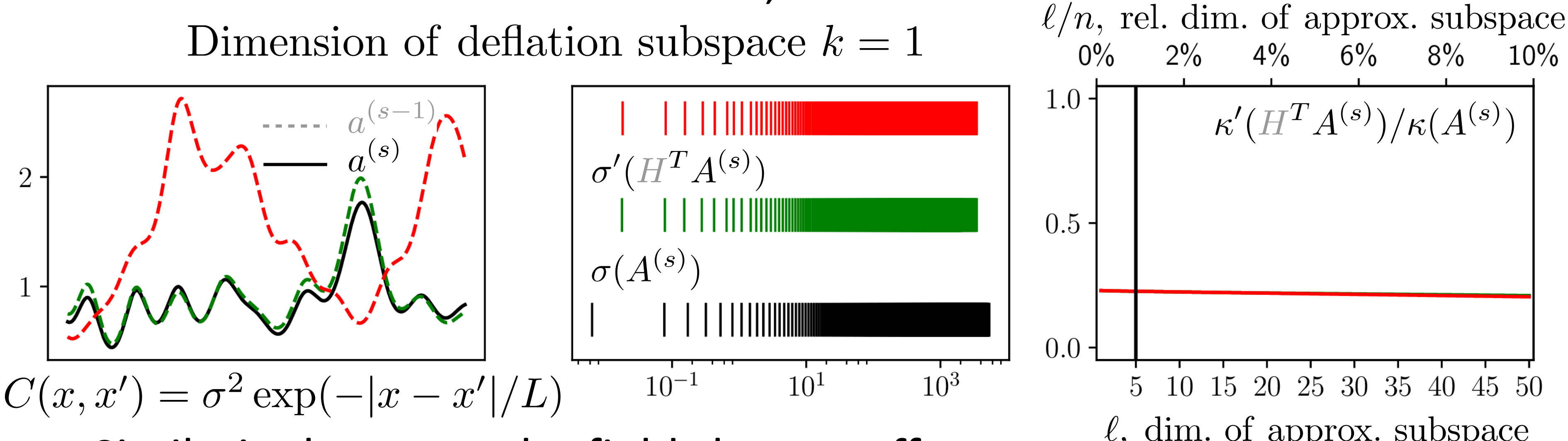
- If a basis $Z^{(s)} \in \mathbb{R}^{n\times m}$ of $\mathcal{S}^{(s)}$ is available, the Ritz vector $w = Z^{(s)}\hat{w}$ is obtained by solving the m-dimensional generalized eigenvalue problem

$$(A^{(s)}Z^{(s)})^T A^{(s)}Z^{(s)}\hat{w} = \theta Z^{(s)T}A^{(s)}Z^{(s)}\hat{w}$$

- Note that upon generating some iterates $\{x_i\}_{i=0}^{\ell}$ to solve $A^{(s-1)}x^{(s-1)} = b$, a basis is constructed for $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$
- Therefore, the system $A^{(s)}x^{(s)} = b$ can be deflated with a basis $W^{(s)}$ made of Harmonic Ritz vectors of $A^{(s)}$ in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$.

Effect of similarity between sampled fields on quality of deflation

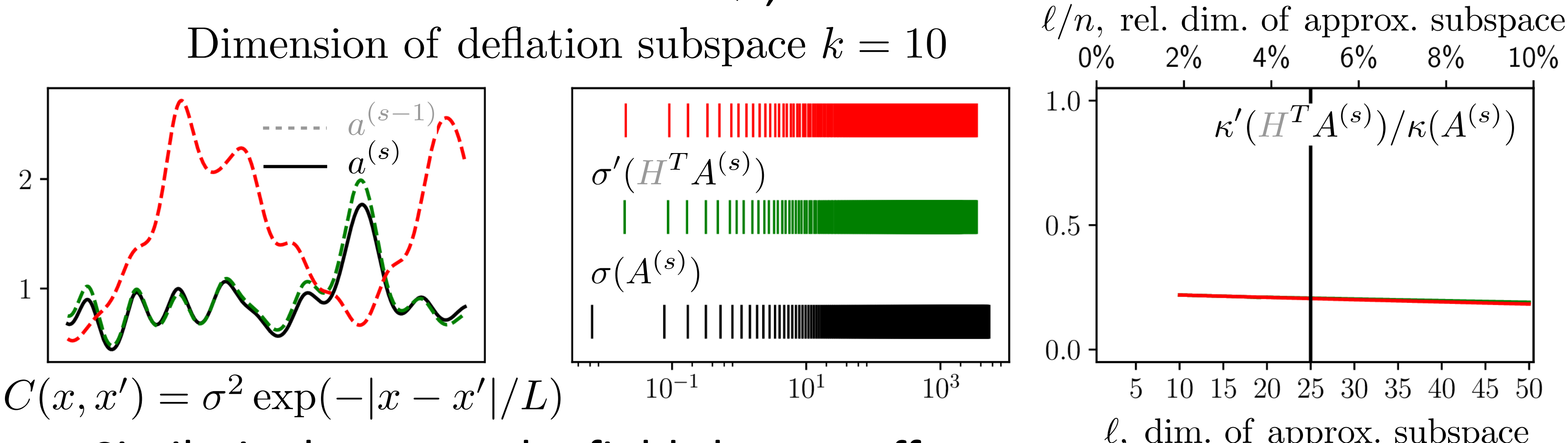
- Deflate $A^{(s)}$ with approximate eigenvectors contained in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$. Effect of relation between $A^{(s-1)}$ and $A^{(s)}$ on the deflation process?
- Let $a^{(s)}$ be the coefficient field of $A^{(s)}$, and consider different fields $a^{(s-1)}$ of $A^{(s-1)}$. For $W^{(s-1)} = \emptyset$, we observe:



- Similarity between the fields has no effect,
- The dimension of the approximation subspace has no effect.

Effect of similarity between sampled fields on quality of deflation

- Deflate $A^{(s)}$ with approximate eigenvectors contained in $\mathcal{R}(W^{(s-1)}) \oplus \mathcal{K}_\ell(A^{(s-1)}, r_0)$. Effect of relation between $A^{(s-1)}$ and $A^{(s)}$ on the deflation process?
- Let $a^{(s)}$ be the coefficient field of $A^{(s)}$, and consider different fields $a^{(s-1)}$ of $A^{(s-1)}$. For $W^{(s-1)} = \emptyset$, we observe:



- Similarity between the fields has no effect,
- The dimension of the approximation subspace has no effect.

DPCG vs PDCG

- Consider a preconditioner $M = LL^T$. Let $\dot{A} := L^{-1}AL^{-T}$ and define the projector $\dot{H} = I_n - \dot{W}(\dot{W}^T\dot{A}\dot{W})^{-1}\dot{W}^T\dot{A}$ with $k \ll n$ linearly independent vectors $[\dot{w}_1, \ldots, \dot{w}_k] =: \dot{W}$.
- A sequence $\{x_i\}_{i=0}^{j}$ of approximations of $x^* := A^{-1}b$ is obtained by

$$x_i := L^{-T}(\dot{W}\hat{\mu}^* + \dot{H}x_i') \quad \text{where}$$

x_i' is an iterative solution of the *deflated system*

$$\dot{H}^T\dot{A}x' = \dot{H}^T L^{-1}b$$

- Error bounded by $\|x^* - x_i\|_A \leq 2\|x^* - x_0\|_A \left(\frac{\sqrt{\kappa'(\dot{H}^T\dot{A})} - 1}{\sqrt{\kappa'(\dot{H}^T\dot{A})} + 1}\right)^i$
- Let $W := L^{-T}\dot{W}$ so that

$$\dot{H}^T\dot{A} = L^{-1}H^TAL^{-T}$$

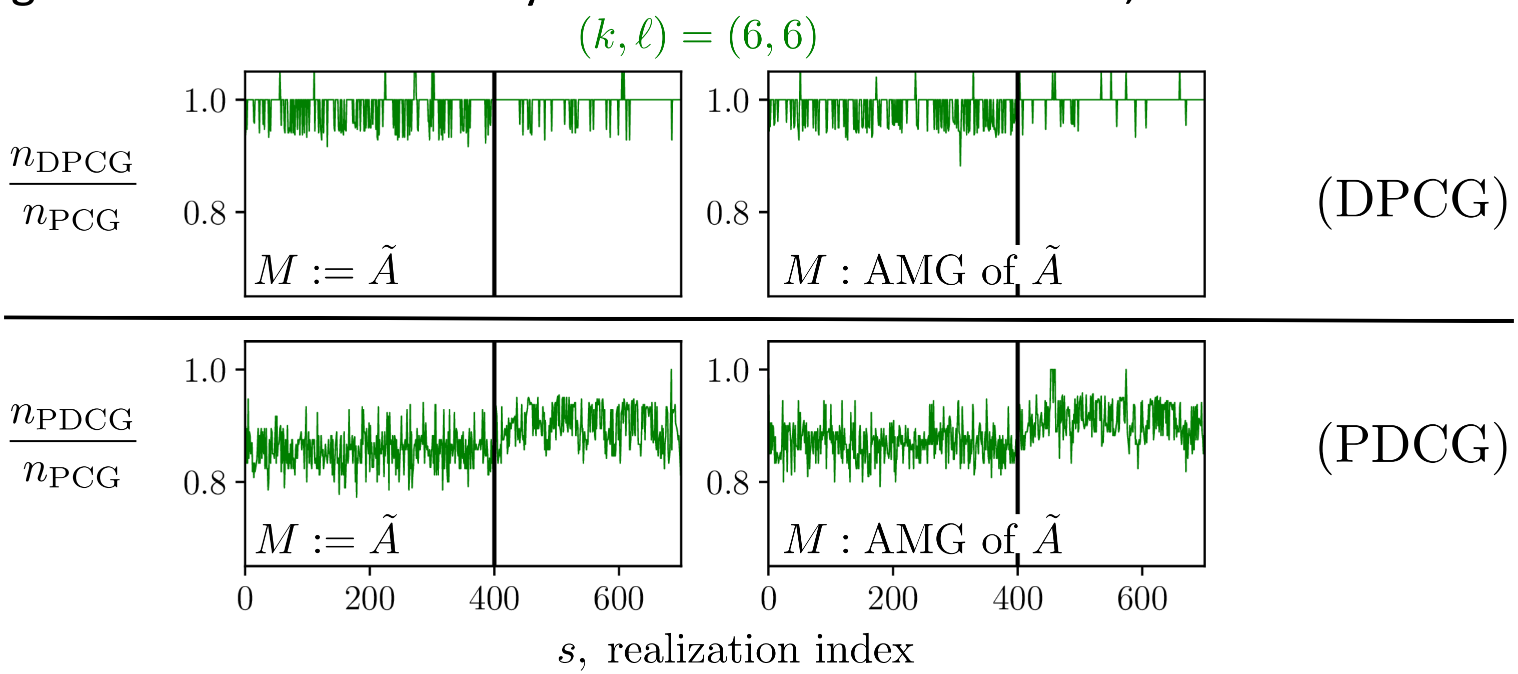
- Two approaches :

(1) Let $\dot{W}$ be eigenvectors of $\dot{A}$ $\Longrightarrow$ W are eigenvectors of $M^{-1}A$ — DPCG

(2) Let W be eigenvectors of A $\Longrightarrow$ $\dot{W}$ are eigenvectors of L^TAL^{-T} — PDCG

Solving a sampled sequence of systems by DPCG or PDCG with approximate eigenvectors

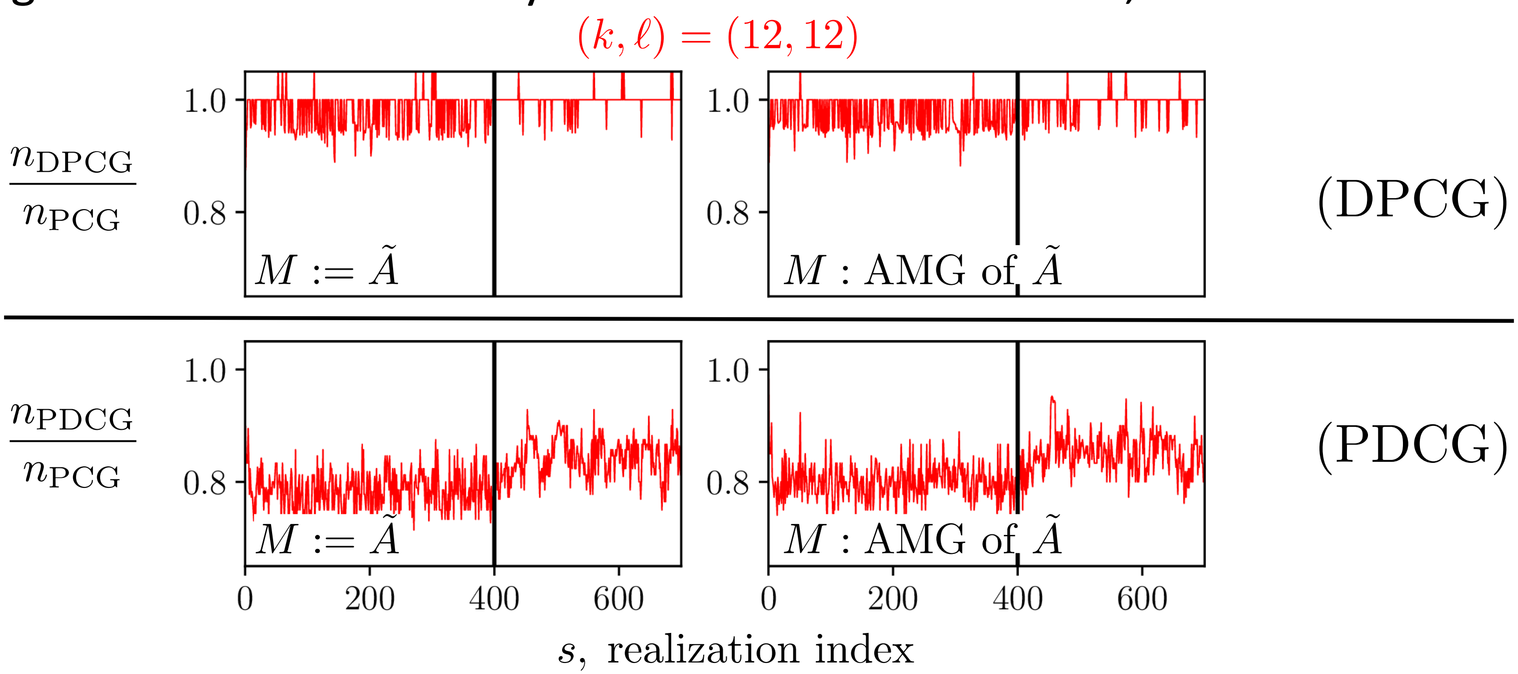
- Deflate current system with approximate eigenvectors $\dot{W}^{(s)} := [\dot{w}_1^{(s)}, \ldots, \dot{w}_k^{(s)}]$ of $L^{-1}A^{(s)}L^{-T}$ (DPCG), or $L^{T}A^{(s)}L^{-T}$ (PDCG), obtained by harmonic Ritz analysis in an approximation subspace of dimension $(k+\ell)$ obtained while solving the precedent system $A^{(s-1)}x^{(s-1)} = b$.
- Sampling the coefficient fields by Markov chain Monte Carlo, we obtain:



Interrupting the deflation subspace update while sampling by MCMC causes a slight deterioration in the quality of the deflation.

Solving a sampled sequence of systems by DPCG or PDCG with approximate eigenvectors

- Deflate current system with approximate eigenvectors $\dot{W}^{(s)} := [\dot{w}_1^{(s)}, \ldots, \dot{w}_k^{(s)}]$ of $L^{-1}A^{(s)}L^{-T}$ (DPCG), or $L^T A^{(s)} L^{-T}$ (PDCG), obtained by harmonic Ritz analysis in an approximation subspace of dimension $(k+\ell)$ obtained while solving the precedent system $A^{(s-1)}x^{(s-1)} = b$.
- Sampling the coefficient fields by Markov chain Monte Carlo, we obtain:



Interrupting the deflation subspace update while sampling by MCMC causes a slight deterioration in the quality of the deflation.